

# THE ROLE OF SPECTRAL ANISOTROPY IN THE RESOLUTION OF THE THREE-DIMENSIONAL NAVIER-STOKES EQUATIONS

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ABSTRACT. We present different classes of initial data to the three-dimensional, incompressible Navier-Stokes equations, which generate a global in time, unique solution though they may be arbitrarily large in the end-point function space in which a fixed-point argument may be used to solve the equation locally in time. The main feature of these initial data is an anisotropic distribution of their frequencies. One of those classes is taken from [5]-[6], and another one is new.

## 1. INTRODUCTION

In this article, we are interested in the construction of global smooth solutions which cannot be obtained in the framework of small data. Let us recall what the incompressible Navier-Stokes (with constant density) is:

$$(NS) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \Delta u = -\nabla p & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0. \end{cases}$$

In all this paper  $x = (x_h, x_3) = (x_1, x_2, x_3)$  will denote a generic point of  $\mathbb{R}^3$  and we shall write  $u = (u^h, u^3) = (u^1, u^2, u^3)$  for a vector field on  $\mathbb{R}^3 = \mathbb{R}_h^2 \times \mathbb{R}_v$ . We also define the horizontal differentiation operators  $\nabla^h \stackrel{\text{def}}{=} (\partial_1, \partial_2)$  and  $\operatorname{div}_h \stackrel{\text{def}}{=} \nabla^h \cdot$ , as well as  $\Delta_h \stackrel{\text{def}}{=} \partial_1^2 + \partial_2^2$ .

First, let us recall the history of global existence results for small data. In his seminal work [15], J. Leray proved in 1934 that if  $\|u_0\|_{L^2} \|\nabla u_0\|_{L^2}$  is small enough, then there exists a global regular solution of  $(NS)$ . Then in [8], H. Fujita and T. Kato proved in 1964 that if

$$\|u_0\|_{\dot{H}^{\frac{1}{2}}} \stackrel{\text{def}}{=} \left( \int_{\mathbb{R}^3} |\xi| |\widehat{u}_0(\xi)|^2 d\xi \right)^{\frac{1}{2}}$$

is small enough, then there exists a unique global solution in the space

$$C_b(\mathbb{R}^+; \dot{H}^{\frac{1}{2}}) \cap L^4(\mathbb{R}^+; \dot{H}^1).$$

After works of many authors on this question (see in particular [11], [13], [17], and [3]), the optimal norm to express the smallness of the initial data was found on 2001 by H. Koch and D. Tataru in [14]. This is the  $BMO^{-1}$  norm. We are not going to define precisely this norm here. Let us simply notice that this norm is in between two Besov norms which can be easily defined. More precisely we have

$$\|u_0\|_{\dot{B}_{\infty,\infty}^{-1}} \lesssim \|u_0\|_{BMO^{-1}} \lesssim \|u_0\|_{\dot{B}_{\infty,2}^{-1}} \quad \text{with}$$

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$$\|u_0\|_{\dot{B}_{\infty,\infty}^{-1}} \stackrel{\text{def}}{=} \sup_{t>0} t^{\frac{1}{2}} \|e^{t\Delta} u_0\|_{L^\infty} \quad \text{and} \quad \|u_0\|_{\dot{B}_{\infty,2}^{-1}} \stackrel{\text{def}}{=} \|e^{t\Delta} u_0\|_{L^2(\mathbb{R}^+; L^\infty)}.$$

First of all, let us mention that  $\dot{H}^{\frac{1}{2}}$  is continuously embedded in  $\dot{B}_{\infty,2}^{-1}$ . To have a more precise idea of what these spaces mean, let us observe that the space  $\dot{B}_{\infty,\infty}^{-1}$  we shall denote by  $\dot{C}^{-1}$  from now on, contains all the derivatives of order 1 of bounded functions. Let us give some examples. If we consider a divergence free vector field of the type

$$u_{\varepsilon,0}(x) = \frac{1}{\varepsilon} \cos\left(\frac{x_3}{\varepsilon}\right) (-\partial_2 \phi(x), \partial_1 \phi(x), 0)$$

for some given function  $\phi$  in the Schwartz class of  $\mathbb{R}^3$ , then we have

$$\|u_{\varepsilon,0}\|_{\dot{B}_{\infty,2}^{-1}} \sim \|u_{\varepsilon,0}\|_{\dot{C}^{-1}} \sim 1 \quad \text{and} \quad \|u_{\varepsilon,0}\|_{\dot{H}^{\frac{1}{2}}} \sim \varepsilon^{-\frac{3}{2}}.$$

Another example which will be a great interest for this paper is the case when

$$u_{\varepsilon,0}(x) = \phi_0(\varepsilon x_3) (-\partial_2 \phi(x_h), \partial_1 \phi(x_h), 0).$$

As claimed by Proposition 1.1 of [5], we have, for small enough  $\varepsilon$ ,

$$(1.1) \quad \|u_{\varepsilon,0}\|_{\dot{C}^{-1}} \geq \frac{1}{2} \|\phi\|_{\dot{C}^{-1}(\mathbb{R}^2)} \|\phi_0\|_{L^\infty(\mathbb{R})}.$$

In this paper, we are going to consider initial data the regularity of which will be (at least)  $\dot{H}^{\frac{1}{2}}$ . Our interest is focused on the size of the initial data measured in the  $\dot{C}^{-1}$  norm.

Let us define  $\mathcal{G}$  the set of divergence free vector fields in  $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$  generating global smooth solutions to  $(NS)$  and let us recall some known results about the geometry of  $\mathcal{G}$ .

First of all, Fujita-Kato' theorem [8] can be interpreted as follows: the set  $\mathcal{G}$  contains a ball of positive radius. Next let us assume that  $\mathcal{G}$  is not the whole space  $\dot{H}^{\frac{1}{2}}$  (in other words, we assume that an initial data exists which generates singularities in finite time). Then there exists a critical radius  $\rho_c$  such that if  $u_0$  is an initial data such that  $\|u_0\|_{\dot{H}^{\frac{1}{2}}} < \rho_c$ , then  $u_0$  generates a global regular solution and for any  $\rho > \rho_c$ , there exists an initial data of  $\dot{H}^{\frac{1}{2}}$  norm  $\rho$  which generates a singularity at finite time. Using the theory of profiles introduced in the context of Navier-Stokes equations by the second author (see [9]), W. Rusin and V. Sverak prove in [16] that the set (where  $\mathcal{G}^c$  denotes the complement of  $\mathcal{G}$  in  $\dot{H}^{\frac{1}{2}}$ )

$$\mathcal{G}^c \cap \{u_0 \in \dot{H}^{\frac{1}{2}} / \|u_0\|_{\dot{H}^{\frac{1}{2}}} = \rho_c\}$$

is non empty and compact up to dilations and translations.

In collaboration with P. Zhang, the first two authors prove in [6] that any point  $u_0$  of  $\mathcal{G}$ , is at the center of an interval  $I$  included in  $\mathcal{G}$  and such that the length of  $I$  measured in the  $\dot{C}^{-1}$  norm is arbitrary large. In other words for any  $u_0$  in  $\mathcal{G}$ , there exist arbitrary large (in  $\dot{C}^{-1}$ ) perturbations of this initial data that generate global solutions. As we shall see, the perturbations are strongly anisotropic.

Our aim is to give a new point of view about the important role played by anisotropy in the resolution of the Cauchy problem for  $(NS)$ .

The first result we shall present shows that as soon as enough anisotropy is present in the initial data (where the degree of anisotropy is given by the norm of the data only), then it generates a global unique solution. A similar result can be found in [2, Theorem 1].

**Theorem 1.** *A constant  $c_0$  exists which satisfies the following. If  $(u_{\varepsilon,0})_{\varepsilon>0}$  is a family of divergence free vector field in  $\dot{H}^{\frac{1}{2}}$  such that  $\|u_{\varepsilon,0}\|_{\dot{H}^{\frac{1}{2}}} \leq \rho$  and satisfying*

$$(1.2) \quad \forall \xi \in \text{Supp } \widehat{u}_{\varepsilon,0}, \quad \text{either } |\xi_h| \leq \varepsilon |\xi_3| \quad \text{or} \quad |\xi_3| \leq \varepsilon |\xi_h|,$$

*then, if  $\varepsilon^4 \|u_{\varepsilon,0}\|_{\dot{H}^{\frac{1}{2}}}$  is less than  $c_0$ ,  $u_{\varepsilon,0}$  belongs to  $\mathcal{G}$ .*

Let us remark that this result has little to do with the precise structure of the equations: as will appear clearly in its proof in Section 2, it can actually easily be recast as a small data theorem, the smallness being measured in anisotropic Sobolev spaces. It is therefore of a different nature than the next Theorems 2 and 3, whose proofs on the contrary rely heavily on the structure of the nonlinearity (more precisely on the fact that the two-dimensional equations are globally well-posed).

The next theorem shows that as soon as the initial data has slow variations in one direction, then it generates a global solution, which, roughly speaking, corresponds to the case when the support in Fourier space of the initial data lies in the region where  $|\xi_3| \leq \varepsilon |\xi_h|$ . Furthermore, one can add to any initial data in  $\mathcal{G}$  any such slowly varying data, and the superposition still generates a global solution (provided the variation is slow enough and the profile vanishes at zero).

**Theorem 2** ([5],[6]). *Let  $v_0^h = (v_0^1, v_0^2)$  be a horizontal, smooth divergence free vector field on  $\mathbb{R}^3$  (i.e.  $v_0^h$  is in  $L^2(\mathbb{R}^3)$  as well as all its derivatives), belonging, as well as all its derivatives, to  $L^2(\mathbb{R}_{x_3}; \dot{H}^{-1}(\mathbb{R}^2))$ ; let  $w_0$  be a smooth divergence free vector field on  $\mathbb{R}^3$ . Then, there exists a positive  $\varepsilon_0$  depending on norms of  $v_0^h$  and  $w_0$  such that, if  $\varepsilon \leq \varepsilon_0$ , then the following initial data belongs to  $\mathcal{G}$  :*

$$v_{\varepsilon,0}(x) \stackrel{\text{def}}{=} (v_0^h + \varepsilon w_0^h, w_0^3)(x_1, x_2, \varepsilon x_3).$$

*If moreover  $v_0^h(x_1, x_2, 0) = w_0^3(x_1, x_2, 0) = 0$  for all  $(x_1, x_2) \in \mathbb{R}^2$ , and if  $u_0$  belongs to  $\mathcal{G}$ , then there exists a positive number  $\varepsilon'_0$  depending on  $u_0$  and on norms of  $v_0^h$  and  $w_0$  such that if  $\varepsilon \leq \varepsilon'_0$ , the following initial data belongs to  $\mathcal{G}$  :*

$$u_{\varepsilon,0} \stackrel{\text{def}}{=} u_0 + v_{\varepsilon,0}.$$

One can assume that  $v_0^h$  and  $w_0^3$  have frequency supports in a given ring of  $\mathbb{R}^3$ , so that (1.2) holds. Nevertheless Theorem 1 not apply since  $v_{\varepsilon,0}$  is of the order of  $\varepsilon^{-\frac{1}{2}}$  in  $\dot{H}^{\frac{1}{2}}$ . Actually the proof of Theorem 2 is deeper than that of Theorem 1, as it uses the structure of the quadratic term in (NS). The proof of Theorem 2 may be found in [5] and [6], we shall not give it here. Note that Inequality (1.1) implies that  $v_{\varepsilon,0}$  may be chosen arbitrarily large in  $\dot{C}^{-1}$ .

One formal way to translate the above result is that the vertical frequencies of the initial data  $v_{\varepsilon,0}$  are actually very small, compared with the horizontal frequencies. The following theorem gives a statement in terms of frequency sizes, in the spirit of Theorem 1. However as already pointed out, Theorem 1 again does not apply because the initial data is too large in  $\dot{H}^{\frac{1}{2}}$ . Notice also that the assumption made in the statement of Theorem 2 that the profile should vanish at  $x_3 = 0$  is replaced here by a smallness assumption in  $L^2(\mathbb{R}^2)$ .

**Theorem 3.** *Let  $(v_{\varepsilon,0})_{\varepsilon}$  be a family of smooth divergence free vector field, uniformly bounded in the space  $L^\infty(\mathbb{R}; \dot{H}^s(\mathbb{R}^2))$  for all  $s \geq -1$ , such that  $(\sqrt{\varepsilon} v_{\varepsilon,0})_{\varepsilon}$  is uniformly bounded in the*

space  $L^2(\mathbb{R}_{x_3}; \dot{H}^s(\mathbb{R}^2))$  for  $s \geq -1$ , and satisfying

$$\forall \varepsilon \in ]0, 1[, \quad \forall \xi \in \text{Supp } \widehat{v}_{\varepsilon,0}, \quad |\xi_3| \leq \varepsilon |\xi_h|.$$

Then there exists a positive number  $\varepsilon_0$  such that for all  $\varepsilon \leq \varepsilon_0$ , the data  $v_{\varepsilon,0}$  belongs to  $\mathcal{G}$ .

Moreover if  $u_0$  belongs to  $\mathcal{G}$ , then there are positive constants  $c_0$  and  $\varepsilon'_0$  such that if

$$\|v_{\varepsilon,0}(\cdot, 0)\|_{L^2(\mathbb{R}_h^2)} \leq c_0$$

then for all  $\varepsilon \leq \varepsilon'_0$ , the following initial data belongs to  $\mathcal{G}$  :

$$u_{\varepsilon,0} \stackrel{\text{def}}{=} u_0 + v_{\varepsilon,0}.$$

Let us remark that as in [5], the data  $v_{\varepsilon,0}$  may be arbitrarily large in  $\dot{C}^{-1}$ . Note that Theorems 2 and 3, though of similar type, are not comparable (unless one imposes the spectrum of the initial profiles in Theorem 2 to be included in a ring of  $\mathbb{R}^3$ , in which case the result follows from Theorem 3).

The paper is organized as follows. In the second section, we introduce anisotropic Sobolev spaces and as a warm up, we prove Theorem 1.

The rest of the paper is devoted to the proof of Theorem 3. In the thid section, we define a (global) approximated solution and prove estimates on this approximated solutions and prove Theorem 3.

The last section is devoted to the proof of a propagation result for a linear transport diffusion equation we admit in the preceeding section. Let us point out that we make the choice not to use the technology anisotropic paradifferential calculus and to present an elementary proof.

## 2. PRELIMINARIES: NOTATION AND ANISOTROPIC FUNCTION SPACES

In this section we recall the definition of the various function spaces we shall be using in this paper, namely anisotropic Lebesgue and Sobolev spaces.

We denote by  $L_h^p L_v^q$  (resp.  $L_v^q(L_h^p)$ ) the space  $L^p(\mathbb{R}_h^2; L^q(\mathbb{R}_v))$  (resp.  $L^q(\mathbb{R}_v; L^p(\mathbb{R}_h^2))$ ) equipped with the norm

$$\|f\|_{L_h^p L_v^q} \stackrel{\text{def}}{=} \left( \int_{\mathbb{R}_h^2} \left( \int_{\mathbb{R}_v} |f(x_h, x_3)|^q dx_3 \right)^{\frac{p}{q}} dx_h \right)^{\frac{1}{p}}$$

and similarly  $\dot{H}^{s,\sigma}$  is the space  $\dot{H}^s(\mathbb{R}^2; \dot{H}^\sigma(\mathbb{R}))$  with

$$\|f\|_{\dot{H}^{s,\sigma}} \stackrel{\text{def}}{=} \left( \int_{\mathbb{R}^3} |\xi_h|^{2s} |\xi_3|^{2\sigma} |\widehat{f}(\xi_h, \xi_3)|^2 d\xi_h d\xi_3 \right)^{\frac{1}{2}}$$

where  $\widehat{f} = \mathcal{F}f$  is the Fourier transform of  $f$ . Note that  $\dot{H}^{s,\sigma}$  is a Hilbert space as soon as  $s < 1$  and  $\sigma < 1/2$ . We define also

$$\|f\|_{\dot{H}^{s_1, s_2, s_3}} \stackrel{\text{def}}{=} \left( \int_{\mathbb{R}^3} |\xi_1|^{2s_1} |\xi_2|^{2s_2} |\xi_3|^{2s_3} |\widehat{f}(\xi_1, \xi_2, \xi_3)|^2 d\xi_1 d\xi_2 d\xi_3 \right)^{\frac{1}{2}}.$$

This is a Hilbert space if all  $s_j$  are less than  $1/2$ . Finally we shall often use the spaces  $L_v^p \dot{H}_h^s = L^p(\mathbb{R}_v; \dot{H}_h^s(\mathbb{R}_h^2))$ . Let us notice that  $L_v^2 \dot{H}_h^s = \dot{H}^{s,0}$ . The following result, proved by D. Iftimie in [12] is the basis of the proof of Theorem 1.

**Theorem 4.** *There is a constant  $\varepsilon_0$  such that the following result holds. Let  $(s_i)_{1 \leq i \leq 3}$  be such that  $s_1 + s_2 + s_3 = 1/2$  and  $-1/2 < s_i < 1/2$ . Then any divergence free vector field of norm smaller than  $\varepsilon_0$  in  $\dot{H}^{s_1, s_2, s_3}$  generates a global smooth solution to (NS).*

This theorem implies obviously the following corollary, since  $\dot{H}^{s, \frac{1}{2}-s}$  is continuously embedded in  $\dot{H}^{\frac{s}{2}, \frac{s}{2}, \frac{1}{2}-s}$  as soon as  $0 < s < 1/2$ . More precisely, we have that the space  $\dot{H}^{s, \frac{1}{2}-s}$  is the space  $\dot{H}^{s, 0, \frac{1}{2}-s} \cap \dot{H}^{0, s, \frac{1}{2}-s}$ .

**Corollary 2.1.** *There is a constant  $\varepsilon_0$  such that the following result holds. Let  $s$  be given in  $]0, 1/2[$ . Then any divergence free vector field of norm smaller than  $\varepsilon_0$  in  $\dot{H}^{s, \frac{1}{2}-s}$  generates a global smooth solution to (NS).*

*Proof of Theorem 1.* Let us decompose  $u_0$  into two parts, namely we write  $u_0 = v_0 + w_0$ , with

$$v_0 \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{|\xi_h| \leq \varepsilon |\xi_3|} \widehat{u}_0(\xi)) \quad \text{and} \quad w_0 \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{|\xi_3| \leq \varepsilon |\xi_h|} \widehat{u}_0(\xi)).$$

Let  $0 < s < 1/2$  be given. On the one hand we have

$$\|v_0\|_{\dot{H}^{s, \frac{1}{2}-s}}^2 = \int_{|\xi_3| \leq \varepsilon |\xi_h|} |\xi_h|^{2s} |\xi_3|^{1-2s} |\widehat{u}_0(\xi)|^2 d\xi$$

hence since  $s < 1/2$ ,

$$\begin{aligned} \|v_0\|_{\dot{H}^{s, \frac{1}{2}-s}}^2 &\leq \varepsilon^{1-2s} \int |\xi_h| |\widehat{u}_0(\xi)|^2 d\xi \\ &\leq \varepsilon^{1-2s} \|u_0\|_{\dot{H}^{\frac{1}{2}}}^2. \end{aligned}$$

Identical computations give, since  $s > 0$ ,

$$\begin{aligned} \|w_0\|_{\dot{H}^{s, \frac{1}{2}-s}}^2 &= \int_{|\xi_h| \leq \varepsilon |\xi_3|} |\xi_h|^{2s} |\xi_3|^{1-2s} |\widehat{u}_0(\xi)|^2 d\xi \\ &\leq \varepsilon^{2s} \int |\xi_3| |\widehat{u}_0(\xi)|^2 d\xi \\ &\leq \varepsilon^{2s} \|u_0\|_{\dot{H}^{\frac{1}{2}}}^2. \end{aligned}$$

To conclude we can choose  $s = 1/4$ , which gives

$$\|u_0\|_{\dot{H}^{\frac{1}{4}, \frac{1}{4}}} \leq \varepsilon^{\frac{1}{4}} \|u_0\|_{\dot{H}^{\frac{1}{2}}}.$$

Then, the result follows by the wellposedness of (NS) in  $\dot{H}^{\frac{1}{4}, \frac{1}{4}}$  given by Corollary 2.1.  $\square$

**Remark 2.1.** The proof of Theorem 1 does not use the special structure of the nonlinear term in (NS) as it reduces to checking that the initial data is small in an adequate scale-invariant space.

### 3. PROOF OF THEOREM 3

In this section we shall prove the second part of Theorem 3: we consider an initial data  $u_0 + v_{\varepsilon,0}$  satisfying the assumptions of the theorem and we prove that for  $\varepsilon > 0$  small enough, it generates a global, unique solution to (NS). It will be clear from the proof that in the case when  $u_0 \equiv 0$  (which amounts to the first part of Theorem 3), the assumption that  $v_{\varepsilon,0}(x_h, 0)$  is small in  $L^2(\mathbb{R}^2)$  is not necessary. Thus the proof of the whole of Theorem 3 will be obtained.

**3.1. Decomposition of the initial data.** The first step of the proof consists in decomposing the initial data as follows.

**Proposition 3.1.** *Let  $v_{\varepsilon,0}$  be a divergence free vector field satisfying*

$$\forall \varepsilon \in ]0, 1[, \quad \forall \xi \in \text{Supp } \widehat{v}_{\varepsilon,0}, \quad |\xi_3| \leq \varepsilon |\xi_h|.$$

*Then there exist two divergence free vector fields  $(\overline{v}_{\varepsilon,0}^h, 0)$  and  $w_{\varepsilon,0}$  the spectrum of which is included in that of  $v_{\varepsilon,0}$ , and such that*

$$v_{\varepsilon,0} = (\overline{v}_{\varepsilon,0}^h, 0) + w_{\varepsilon,0} \quad \text{with} \quad |\widehat{w}_{\varepsilon,0}^h| \leq \varepsilon |\widehat{w}_{\varepsilon,0}^3|.$$

*Proof.* Let  $\mathbb{P}_h \stackrel{\text{def}}{=} \text{Id} - \nabla_h \Delta_h^{-1} \text{div}_h$  be the Leray projector onto horizontal divergence free vector fields and define

$$(3.3) \quad \overline{v}_{\varepsilon,0}^h \stackrel{\text{def}}{=} \mathbb{P}_h v_{\varepsilon,0}^h \quad \text{and} \quad w_{\varepsilon,0} \stackrel{\text{def}}{=} v_{\varepsilon,0} - (\overline{v}_{\varepsilon,0}^h, 0).$$

The estimate on  $w_{\varepsilon,0}$  simply comes from the fact that obviously

$$\widehat{w}_{\varepsilon,0}^h(\xi) = \frac{\xi_h \cdot \widehat{v}_{\varepsilon,0}^h}{|\xi_h|^2} \xi_h,$$

and therefore since  $v_{\varepsilon,0}$  is divergence free and using the spectral assumption we find

$$|\widehat{w}_{\varepsilon,0}^h(\xi)| \leq \frac{|\xi_h \cdot \widehat{v}_{\varepsilon,0}^h|}{|\xi_h|} = \frac{|\xi_3 \widehat{v}_{\varepsilon,0}^3|}{|\xi_h|} \leq \varepsilon |\widehat{v}_{\varepsilon,0}^3| = \varepsilon |\widehat{w}_{\varepsilon,0}^3(\xi)|.$$

That proves the proposition.  $\square$

### 3.2. Construction of an approximate solution and end of the proof of Theorem 3.

The construction of the approximate solution follows closely the ideas of [5]-[6]. We write indeed

$$v_{\varepsilon}^{\text{app}} \stackrel{\text{def}}{=} (\overline{v}_{\varepsilon}^h, 0) + w_{\varepsilon} \quad \text{and} \quad u_{\varepsilon}^{\text{app}} \stackrel{\text{def}}{=} u + v_{\varepsilon}^{\text{app}},$$

where  $u$  is the global unique solution associated with  $u_0$  and  $\overline{v}_{\varepsilon}^h$  solves the two dimensional Navier-Stokes equations for each given  $x_3$ :

$$(NS2D)_{x_3} \begin{cases} \partial_t \overline{v}_{\varepsilon}^h + \overline{v}_{\varepsilon}^h \cdot \nabla^h \overline{v}_{\varepsilon}^h - \Delta_h \overline{v}_{\varepsilon}^h = -\nabla^h \overline{p}_{\varepsilon} & \text{in } \mathbb{R}^+ \times \mathbb{R}^2 \\ \text{div}_h \overline{v}_{\varepsilon}^h = 0 \\ \overline{v}_{\varepsilon}^h|_{t=0} = \overline{v}_{\varepsilon,0}^h(\cdot, x_3), \end{cases}$$

while  $w_{\varepsilon}$  solves the linear transport-diffusion type equation

$$(T) \begin{cases} \partial_t w_{\varepsilon} + \overline{v}_{\varepsilon}^h \cdot \nabla^h w_{\varepsilon} - \Delta w_{\varepsilon} = -\nabla q_{\varepsilon} & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \text{div } w_{\varepsilon} = 0 \\ w_{\varepsilon}|_{t=0} = w_{\varepsilon,0}. \end{cases}$$

Those vector fields satisfy the following bounds (see Paragraph 3.3 for a proof).

**Lemma 3.1.** *Under the assumptions of Theorem 3, the family  $u_\varepsilon^{app}$  is uniformly bounded in  $L^2(\mathbb{R}^+; L^\infty(\mathbb{R}^3))$ , and  $\nabla u_\varepsilon^{app}$  is uniformly bounded in  $L^2(\mathbb{R}^+; L_v^\infty L_h^2)$ .*

Now define  $u_\varepsilon$  the solution associated with the initial data  $u_0 + v_{\varepsilon,0}$ , which a priori has a finite life span, depending on  $\varepsilon$ . Consider

$$R_\varepsilon \stackrel{\text{def}}{=} u_\varepsilon - u_\varepsilon^{app},$$

which satisfies the following property (see Paragraph 3.4 for a proof).

**Lemma 3.2.** *For any positive  $\delta$  there exists  $\varepsilon(\delta)$  and  $c(\delta)$  such that if*

$$\varepsilon \leq \varepsilon(\delta) \quad \text{and if} \quad \|v_{\varepsilon,0}(\cdot, 0)\|_{L_h^2} \leq c(\delta),$$

*then the vector field  $R_\varepsilon \stackrel{\text{def}}{=} u_\varepsilon - u_\varepsilon^{app}$  satisfies the equation*

$$(E_\varepsilon) \quad \begin{cases} \partial_t R_\varepsilon + R_\varepsilon \cdot \nabla R_\varepsilon - \Delta R_\varepsilon + u_\varepsilon^{app} \cdot \nabla R_\varepsilon + R_\varepsilon \cdot \nabla u_\varepsilon^{app} = F_\varepsilon - \nabla \tilde{q}_\varepsilon \\ \operatorname{div} R_\varepsilon = 0 \\ R_\varepsilon|_{t=0} = 0 \end{cases}$$

*with  $\|F_\varepsilon\|_{L^2(\mathbb{R}^+; \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \leq \delta$ .*

Assuming those two lemmas to be true, the end of the proof of Theorem 3 follows very easily using the method given in [5, Section 2]: an energy estimate in  $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$  on  $(E_\varepsilon)$ , using the fact that the forcing term is as small as needed and that the initial data is zero, gives that  $R_\varepsilon$  is unique, and uniformly bounded in  $L^\infty(\mathbb{R}^+; \dot{H}^{\frac{1}{2}}) \cap L^2(\mathbb{R}^+; \dot{H}^{\frac{3}{2}})$ . Since the approximate solution is also unique and globally defined, Theorem 3 is proved.  $\square$

**3.3. Proof of the estimates on the approximate solution (Lemma 3.1).** As noted in [6, Appendix B], the global solution  $u$  associated with  $u_0 \in \dot{H}^{\frac{1}{2}}$  belongs to  $L^2(\mathbb{R}^+; L^\infty(\mathbb{R}^3))$ , and  $\nabla u$  belongs to  $L^2(\mathbb{R}^+; L_v^\infty L_h^2)$ . So we just need to study  $v_\varepsilon^{app}$ , which we shall do in two steps: first  $\bar{v}_\varepsilon^h$ , then  $w_\varepsilon$ .

**3.3.1. Estimates on  $\bar{v}_\varepsilon^h$ .** Due to the spectral assumption on  $\bar{v}_{\varepsilon,0}^h$ , it is easy to see that

$$\begin{aligned} \forall \alpha = (\alpha_h, \alpha_3) \in \mathbb{N}^2 \times \mathbb{N}, \quad \varepsilon^{\frac{1}{2}-\alpha_3} \partial^\alpha \bar{v}_{\varepsilon,0}^h \quad \text{is uniformly bounded in } L_v^2 \dot{H}_h^s, \\ \text{and} \quad \varepsilon^{-\alpha_3} \partial^\alpha \bar{v}_{\varepsilon,0}^h \quad \text{is uniformly bounded in } L_v^\infty \dot{H}_h^s. \end{aligned}$$

Indeed the definition of  $\bar{v}_{\varepsilon,0}^h$  given in (3.3), and the spectral assumption as well as the a priori bounds on  $v_{\varepsilon,0}$ , give directly the first result. To prove the second result one uses first the Gagliardo-Nirenberg inequality:

$$\|\partial^\alpha \bar{v}_{\varepsilon,0}^h\|_{L_v^\infty \dot{H}_h^s}^2 \leq \|\partial^\alpha \bar{v}_{\varepsilon,0}^h\|_{L_v^2 \dot{H}_h^s} \|\partial_3 \partial^\alpha \bar{v}_{\varepsilon,0}^h\|_{L_v^2 \dot{H}_h^s},$$

and then the same arguments. The proof of [5, Lemma 3.1 and Corollary 3.1] enables us to infer from those bounds the following result.

**Proposition 3.2.** *Under the assumptions of Theorem 3, for all real numbers  $s > -1$  and all  $\alpha = (\alpha_h, \alpha_3) \in \mathbb{N}^2 \times \mathbb{N}$  there is a constant  $\overline{C}$  such that the vector field  $\overline{v}_\varepsilon^h$  satisfies the following bounds:*

$$\begin{aligned} \|\partial^\alpha \overline{v}_\varepsilon^h(t)\|_{L_v^\infty \dot{H}_h^s}^2 + \sup_{x_3 \in \mathbb{R}} \int_0^t \|\partial^\alpha \nabla^h \overline{v}_\varepsilon^h(t')\|_{\dot{H}_h^s}^2 dt' \\ + \varepsilon \left( \|\partial^\alpha \nabla^h \overline{v}_\varepsilon^h(t)\|_{L_v^2 \dot{H}_h^s}^2 + \int_0^t \|\partial^\alpha \overline{v}_\varepsilon^h(t')\|_{L^2(\mathbb{R}^3)}^2 dt' \right) \leq \overline{C} \varepsilon^{2\alpha_3}. \end{aligned}$$

**3.3.2. Estimates on  $w_\varepsilon$ .** The definition of  $w_{\varepsilon,0}$  given in (3.3), along with the spectral assumption on  $(v_{\varepsilon,0})_{\varepsilon>0}$  lead to

$$\forall \varepsilon \in ]0, 1[, \quad \forall \xi \in \text{Supp } \widehat{w}_{\varepsilon,0}, \quad |\xi_3| \leq \varepsilon |\xi_h| \quad \text{and} \quad |\widehat{w}_{\varepsilon,0}^h(\xi)| \leq \varepsilon |\widehat{w}_{\varepsilon,0}^3(\xi)|.$$

The proof of the following result is technical and postponed to section 4.

**Proposition 3.3.** *Under the assumptions of Theorem 3,  $w_\varepsilon^3$  and  $\varepsilon^{-1}w_\varepsilon^h$  are uniformly bounded in the space  $L^\infty(\mathbb{R}^+; L_v^\infty L_h^2) \cap L^2(\mathbb{R}^+; L_v^\infty \dot{H}_h^s)$  for all  $s \geq 0$ . Moreover  $\varepsilon^{\frac{1}{2}-\alpha_3} \partial^\alpha w_\varepsilon$  is uniformly bounded in  $L^\infty(\mathbb{R}^+; L_v^2 \dot{H}_h^s) \cap L^2(\mathbb{R}^+; L_v^2 \dot{H}_h^s)$  for all  $s \geq 0$  and all  $\alpha = (\alpha_h, \alpha_3) \in \mathbb{N}^2 \times \mathbb{N}$ .*

The Gagliardo-Nirenberg inequality and Sobolev embeddings lead to Lemma 3.1.

**3.4. Proof of the estimates on the remainder (Lemma 3.2).** Subtracting the equation on  $u_\varepsilon^{app}$  from the equation on  $u$  one finds directly that

$$F_\varepsilon = (\partial_3^2 \overline{v}_\varepsilon^h, \partial_3 \overline{p}_\varepsilon) + w_\varepsilon \cdot \nabla v_\varepsilon^{app} + u \cdot \nabla v_\varepsilon^{app} + v_\varepsilon^{app} \cdot \nabla u,$$

which we decompose into  $F_\varepsilon = G_\varepsilon + H_\varepsilon$  with

$$G_\varepsilon \stackrel{\text{def}}{=} (\partial_3^2 \overline{v}_\varepsilon^h, \partial_3 \overline{p}_\varepsilon) + w_\varepsilon \cdot \nabla v_\varepsilon^{app} \quad \text{and} \quad H_\varepsilon \stackrel{\text{def}}{=} u \cdot \nabla v_\varepsilon^{app} + v_\varepsilon^{app} \cdot \nabla u.$$

Lemma 3.2 follows from the two following propositions.

**Proposition 3.4.** *There is a positive constant  $C$  such that for all  $\varepsilon$  in  $]0, 1[$ ,*

$$\|G_\varepsilon\|_{L^2(\mathbb{R}^+; \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \leq C \varepsilon^{\frac{1}{2}}.$$

*Proof.* Let us start by splitting  $G_\varepsilon$  in three parts:  $G_\varepsilon = G_\varepsilon^1 + G_\varepsilon^2 + G_\varepsilon^3$  with

$$G_\varepsilon^1 \stackrel{\text{def}}{=} (\partial_3^2 \overline{v}_\varepsilon^h, 0), \quad G_\varepsilon^2 \stackrel{\text{def}}{=} (0, \partial_3 \overline{p}_\varepsilon), \quad \text{and} \quad G_\varepsilon^3 \stackrel{\text{def}}{=} w_\varepsilon \cdot \nabla v_\varepsilon^{app}.$$

On the one hand we have obviously

$$\|G_\varepsilon^1\|_{L^2(\mathbb{R}^+; \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \leq \|\partial_3^2 \overline{v}_\varepsilon^h\|_{L^2(\mathbb{R}^+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))}.$$

Proposition 3.2 applied with  $\alpha = (0, 1)$ ,  $\alpha = (0, 2)$  and  $\alpha = (\alpha_h, 1)$  with  $|\alpha_h| = 1$  gives

$$\int_0^t \|\partial_3 \overline{v}_\varepsilon(t', \cdot)\|_{L^2}^2 dt' \lesssim \varepsilon \quad \text{and} \quad \int_0^t \|\partial_3 \nabla \overline{v}_\varepsilon(t', \cdot)\|_{L^2}^2 dt' \lesssim \varepsilon.$$

By interpolation, we infer that

$$(3.4) \quad \|G_\varepsilon^1\|_{L^2(\mathbb{R}^+; \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \lesssim \varepsilon^{\frac{1}{2}}.$$



To estimate  $G_\varepsilon^2$  we use the fact that

$$-\Delta_h \bar{p}_\varepsilon = \sum_{j,k=1}^2 \partial_j \partial_k (\bar{v}_\varepsilon^j \bar{v}_\varepsilon^k)$$

and since  $(-\Delta_h)^{-1} \partial_j \partial_k$  is a Fourier multiplier of order 0 for each  $(j, k)$  in  $\{1, 2\}^2$  we get

$$\|G_\varepsilon^2\|_{L^2(\mathbb{R}^+; \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \lesssim \sum_{j,k=1}^2 \|\bar{v}_\varepsilon^j \partial_3 \bar{v}_\varepsilon^k\|_{L^2(\mathbb{R}^+; \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))}.$$

As  $L_v^2 \dot{H}_h^{-\frac{1}{2}} \hookrightarrow \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)$ , we get

$$\|G_\varepsilon^2\|_{L^2(\mathbb{R}^+; \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \lesssim \sum_{j,k=1}^2 \|\bar{v}_\varepsilon^j \partial_3 \bar{v}_\varepsilon^k\|_{L^2(\mathbb{R}^+; L_v^2 \dot{H}_h^{-\frac{1}{2}})}.$$

Using the Sobolev embedding  $L_h^{\frac{4}{3}} \hookrightarrow \dot{H}_h^{-\frac{1}{2}}$  and Hölder's inequality gives

$$\begin{aligned} \|G_\varepsilon^2\|_{L^2(\mathbb{R}^+; \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} &\lesssim \sum_{j,k=1}^2 \|\bar{v}_\varepsilon^j \partial_3 \bar{v}_\varepsilon^k\|_{L^2(\mathbb{R}^+; L_v^2 L_h^{\frac{4}{3}})} \\ &\leq C \|\bar{v}_\varepsilon^h\|_{L^\infty(\mathbb{R}^+; L_v^\infty L_h^{\frac{8}{3}})} \|\partial_3 \bar{v}_\varepsilon^h\|_{L^\infty(\mathbb{R}^+; L_v^\infty L_h^{\frac{8}{3}})} \end{aligned}$$

so the Sobolev embedding  $\dot{H}_h^{\frac{1}{4}} \hookrightarrow L_h^{\frac{8}{3}}$  gives finally

$$\|G_\varepsilon^2\|_{L^2(\mathbb{R}^+; \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \lesssim C \|\bar{v}_\varepsilon^h\|_{L^\infty(\mathbb{R}^+; L_v^\infty \dot{H}_h^{\frac{1}{4}})} \|\partial_3 \bar{v}_\varepsilon^h\|_{L^2(\mathbb{R}^+; L_v^2 \dot{H}_h^{\frac{1}{4}})}.$$

The result follows again from Proposition 3.2: choosing  $s = 1/4$  and  $\alpha = 0$  we get that  $\bar{v}_\varepsilon^h$  is uniformly bounded in  $L^\infty(\mathbb{R}^+; L_v^\infty \dot{H}_h^{\frac{1}{4}})$ , while  $s = -3/4$  and  $\alpha = (\alpha_h, 1)$  with  $|\alpha_h| = 1$  gives

$$\|\partial_3 \bar{v}_\varepsilon^h\|_{L^2(\mathbb{R}^+; L_v^2 \dot{H}_h^{\frac{1}{4}})} \lesssim \varepsilon^{\frac{1}{2}}.$$

We infer finally that

$$(3.5) \quad \|G_\varepsilon^2\|_{L^2(\mathbb{R}^+; \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \lesssim \varepsilon^{\frac{1}{2}}.$$

To end the proof of the proposition let us estimate  $G_\varepsilon^3$ . We simply use two-dimensional product laws, which gives

$$\begin{aligned} \|G_\varepsilon^3\|_{L^2(\mathbb{R}^+; \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} &= \|w_\varepsilon \cdot \nabla v_\varepsilon^{app}\|_{L^2(\mathbb{R}^+; \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \\ &\lesssim \|w_\varepsilon^h\|_{L^\infty(\mathbb{R}^+; L_v^\infty \dot{H}_h^{\frac{1}{4}})} \|\nabla^h v_\varepsilon^{app}\|_{L^\infty(\mathbb{R}^+; L_v^2 \dot{H}_h^{\frac{1}{4}})} \\ &\quad + \|w_\varepsilon^3\|_{L^\infty(\mathbb{R}^+; L_v^\infty \dot{H}_h^{\frac{1}{4}})} \|\partial_3 v_\varepsilon^{app}\|_{L^\infty(\mathbb{R}^+; L_v^2 \dot{H}_h^{\frac{1}{4}})} \lesssim \varepsilon^{\frac{1}{2}}, \end{aligned}$$

due to Propositions 3.2 and 3.3. Together with Inequalities (3.4) and (3.5) that proves Proposition 3.4.  $\square$

**Proposition 3.5.** *Let  $\delta > 0$  be given. There are positive constants  $\varepsilon(\delta)$  and  $c(\delta)$  such that if  $\varepsilon \leq \varepsilon(\delta)$  and if  $\|v_{\varepsilon,0}(\cdot, 0)\|_{L_h^2} \leq c(\delta)$ , then*

$$\|H_\varepsilon\|_{L^2(\mathbb{R}^+; \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \leq \delta.$$

*Proof.* First, we approximate  $H_\varepsilon$ , and then we estimate this approximation.

Using [10, Theorem 2.1] we get

$$\lim_{t \rightarrow \infty} \|u(t, \cdot)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = 0$$

so we can approximate  $u$  in  $L^\infty(\mathbb{R}^+, \dot{H}^{\frac{1}{2}})$ : for all  $\eta > 0$ , there exists an integer  $N$ , real numbers  $(t_j)_{0 \leq j \leq N}$  and smooth, compactly supported, divergence free functions  $(\phi_j)_{1 \leq j \leq N}$  such that

$$\tilde{u}_\eta(t) \stackrel{\text{def}}{=} \sum_{j=1}^N \mathbf{1}_{[t_{j-1}, t_j]}(t) \phi_j$$

is uniformly bounded in  $L^\infty(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}) \cap L^2(\mathbb{R}^+, \dot{H}^{\frac{3}{2}})$  and satisfies

$$(3.6) \quad \|u - \tilde{u}_\eta\|_{L^\infty(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))} \leq \eta.$$

We split  $H_\varepsilon$  into two contributions

$$H_\varepsilon = H_{\varepsilon, \eta} + (\tilde{u}_\eta - u) \cdot \nabla v_\varepsilon^{app} + v_\varepsilon^{app} \cdot \nabla (\tilde{u}_\eta - u)$$

with  $H_{\varepsilon, \eta} \stackrel{\text{def}}{=} \tilde{u}_\eta \cdot \nabla v_\varepsilon^{app} + v_\varepsilon^{app} \cdot \nabla \tilde{u}_\eta$ .

As  $v_\varepsilon^{app}$  and  $\tilde{u}_\eta - u$  are divergence free vector fields,

$$H_\varepsilon - H_{\varepsilon, \eta} = \text{div}((\tilde{u}_\eta - u) \otimes v_\varepsilon^{app} + v_\varepsilon^{app} \otimes (\tilde{u}_\eta - u)).$$

Thanks to [6, Lemma 3.3] we get

$$\|H_\varepsilon - H_{\varepsilon, \eta}\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\tilde{u}_\eta - u\|_{\dot{H}^{\frac{1}{2}}} (\|\nabla^h v_\varepsilon^{app}\|_{L_v^\infty L_h^2} + \|v_\varepsilon^{app}\|_{L^\infty} + \|\partial_3 v_\varepsilon^{app}\|_{L_v^2 \dot{H}_h^{\frac{1}{2}}})$$

and Proposition 3.2 along with (3.6) lead to

$$\|H_\varepsilon - H_{\varepsilon, \eta}\|_{L^2(\mathbb{R}^+, \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \lesssim \eta.$$

It remains to estimate  $H_{\varepsilon, \eta} = \tilde{u}_\eta \cdot \nabla v_\varepsilon^{app} + v_\varepsilon^{app} \cdot \nabla \tilde{u}_\eta$ . By Propositions 3.2 and 3.3 we have

$$\begin{aligned} \|\tilde{u}_\eta^3 \partial_3 v_\varepsilon^{app}\|_{L^2(\mathbb{R}^+, \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} &\lesssim \|\tilde{u}_\eta^3\|_{L^\infty(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))} \|\partial_3 v_\varepsilon^{app}\|_{L^2(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))} \\ &\lesssim \|\tilde{u}_\eta^3\|_{L^\infty(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))} \varepsilon^{\frac{1}{2}}. \end{aligned}$$

Since  $\tilde{u}_\eta$  is uniformly bounded in  $L^\infty(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$ , we infer that

$$\lim_{\varepsilon \rightarrow 0} \|\tilde{u}_\eta^3 \partial_3 v_\varepsilon^{app}\|_{L^2(\mathbb{R}^+, \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} = 0.$$

Lemma 3.4 of [6] claims that

$$\|ab\|_{\dot{H}^{-\frac{1}{2}}} \leq C \|a\|_{L_v^2 \dot{H}_h^{\frac{1}{2}}} \|b(\cdot, 0)\|_{L_h^2} + C \|x_3 a\|_{L^2} \|\partial_3 b\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}}.$$

So we get

$$\|\tilde{u}_\eta^h \cdot \nabla^h v_\varepsilon^{app}\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\tilde{u}_\eta^h\|_{L_v^2 \dot{H}_h^{\frac{1}{2}}} \|\nabla^h v_\varepsilon^{app}(\cdot, 0)\|_{L_h^2} + \|x_3 \tilde{u}_\eta^h\|_{L^2} \|\partial_3 \nabla^h v_\varepsilon^{app}\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}}$$

and

$$\|v_\varepsilon^{app} \cdot \nabla \tilde{u}_\eta\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\nabla \tilde{u}_\eta\|_{L_v^2 \dot{H}_h^{\frac{1}{2}}} \|v_\varepsilon^{app}(\cdot, 0)\|_{L_h^2} + \|x_3 \nabla \tilde{u}_\eta\|_{L^2} \|\partial_3 v_\varepsilon^{app}\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}}.$$

Propositions 3.2 and 3.3 lead to

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^+} \|x_3 \tilde{u}_\eta^h(t)\|_{L^2(\mathbb{R}^3)}^2 \|\partial_3 \nabla^h v_\varepsilon^{app}(t)\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}(\mathbb{R}^3)}^2 dt = 0$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^+} \|x_3 \nabla \tilde{u}_\eta(t)\|_{L^2(\mathbb{R}^3)}^2 \|\partial_3 v_\varepsilon^{app}(t)\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}(\mathbb{R}^3)}^2 dt = 0.$$

Now we recall that  $\tilde{u}_\eta$  is uniformly bounded in  $L^\infty(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}) \cap L^2(\mathbb{R}^+, \dot{H}^{\frac{3}{2}})$ , hence  $\tilde{u}_\eta$  is uniformly bounded in  $L^\infty(\mathbb{R}^+, L_v^2 \dot{H}_h^{\frac{1}{2}})$  and  $\nabla \tilde{u}_\eta$  is uniformly bounded in  $L^2(\mathbb{R}^+, L_v^2 \dot{H}_h^{\frac{1}{2}})$ . So in order to conclude we just have to estimate

$$\|v_\varepsilon^{app}(\cdot, 0)\|_{L^\infty(\mathbb{R}^+, L_h^2(\mathbb{R}^2))} + \|\nabla^h v_\varepsilon^{app}(\cdot, 0)\|_{L^2(\mathbb{R}^+, L_h^2(\mathbb{R}^2))}.$$

This is done in the following proposition, which concludes the proof of Proposition 3.5.  $\square$

**Proposition 3.6.** *For all  $\delta > 0$  there are positive constants  $\varepsilon(\delta)$  and  $c(\delta)$  such that for all  $0 < \varepsilon \leq \varepsilon(\delta)$ , if  $\|u_{\varepsilon,0}(\cdot, 0)\|_{L_h^2} \leq c(\delta)$  then*

$$\|v_\varepsilon^{app}(\cdot, 0)\|_{L^\infty(\mathbb{R}^+, L_h^2(\mathbb{R}^2))} + \|\nabla^h v_\varepsilon^{app}(\cdot, 0)\|_{L^2(\mathbb{R}^+, L_h^2(\mathbb{R}^2))} \leq \delta.$$

*Proof.* First, we estimate  $\bar{v}_\varepsilon^h$  and  $w_\varepsilon^h$ . For all  $\varepsilon > 0$ , an energy estimate in  $L_h^2$  gives

$$(3.7) \quad \frac{1}{2} \|\bar{v}_\varepsilon^h(t, \cdot, 0)\|_{L_h^2}^2 + \int_0^t \|\nabla^h \bar{v}_\varepsilon^h(t', \cdot, 0)\|_{L_h^2}^2 dt' = \frac{1}{2} \|v_{\varepsilon,0}(\cdot, 0)\|_{L_h^2}^2.$$

Then, for all  $\delta > 0$  there is a constant  $c(\delta)$  such that if  $\|v_{\varepsilon,0}(\cdot, 0)\|_{L_h^2} \leq c(\delta)$  then

$$\|\bar{v}_\varepsilon^h(\cdot, 0)\|_{L^\infty(\mathbb{R}^+, L_h^2(\mathbb{R}^2))} + \|\nabla^h \bar{v}_\varepsilon^h(\cdot, 0)\|_{L^2(\mathbb{R}^+, L_h^2(\mathbb{R}^2))} \leq \delta.$$

Moreover, by Proposition 3.3 we have

$$\|w_\varepsilon^h(\cdot, 0)\|_{L^\infty(\mathbb{R}^+, L_h^2)} + \|\nabla^h w_\varepsilon^h(\cdot, 0)\|_{L^2(\mathbb{R}^+, L_h^2)} \lesssim \varepsilon.$$

It remains to estimate  $w_\varepsilon^3$ . According to Proposition 3.3,  $w_\varepsilon$  and  $\nabla^h w_\varepsilon$  are uniformly bounded respectively in  $L^\infty(\mathbb{R}^+, L_v^\infty \dot{H}_h^{-\frac{1}{2}})$  and  $L^2(\mathbb{R}^+, L_v^\infty \dot{H}_h^{-\frac{1}{2}})$ , so we shall get the result by proving that for all  $\delta > 0$  there are positive constants  $\varepsilon(\delta)$  and  $c(\delta)$  such that if  $\varepsilon \leq \varepsilon(\delta)$  and  $\|u_{\varepsilon,0}(\cdot, 0)\|_{L_h^2} \leq c(\delta)$  then

$$\|w_\varepsilon^3(\cdot, 0)\|_{L^\infty(\mathbb{R}^+, \dot{H}_h^{\frac{1}{2}})} + \|\nabla^h w_\varepsilon^3(\cdot, 0)\|_{L^2(\mathbb{R}^+, \dot{H}_h^{\frac{1}{2}})} \leq \delta.$$

Recall that  $w_\varepsilon^3$  satisfies

$$\begin{cases} \partial_t w_\varepsilon^3 + \bar{v}_\varepsilon^h \cdot \nabla^h w_\varepsilon^3 - \Delta_h w_\varepsilon^3 = \partial_3^2 w_\varepsilon^3 - \partial_3 q_\varepsilon \\ w_\varepsilon^3|_{t=0} = w_{\varepsilon,0}^3. \end{cases}$$

Define  $T_\varepsilon \stackrel{\text{def}}{=} \partial_3^2 w_\varepsilon^3 - \partial_3 q_\varepsilon$ . An energy estimate in  $\dot{H}_h^{\frac{1}{2}}$  gives

$$(3.8) \quad \begin{aligned} & \|w_\varepsilon^3(t, 0)\|_{\dot{H}_h^{\frac{1}{2}}}^2 + \int_0^t \|\nabla^h w_\varepsilon^3(t', 0)\|_{\dot{H}_h^{\frac{1}{2}}}^2 dt' \\ & \lesssim \|w_{\varepsilon,0}^3(\cdot, 0)\|_{\dot{H}_h^{\frac{1}{2}}}^2 + \|T_\varepsilon(\cdot, 0)\|_{L^2(\mathbb{R}^+, \dot{H}_h^{-\frac{1}{2}})}^2 + \int_0^t |\langle \bar{v}_\varepsilon^h \cdot \nabla^h w_\varepsilon^3, w_\varepsilon^3 \rangle_{\dot{H}_h^{\frac{1}{2}}}|(t', 0) dt'. \end{aligned}$$

Using [4, Lemma 1.1] we get for each fixed  $x_3$

$$|\langle \bar{v}_\varepsilon^h \cdot \nabla^h w_\varepsilon^3, w_\varepsilon^3 \rangle_{\dot{H}_h^{\frac{1}{2}}}(x_3)| \lesssim \|\nabla^h \bar{v}_\varepsilon^h(x_3)\|_{L_h^2} \|\nabla^h w_\varepsilon^3(x_3)\|_{\dot{H}_h^{\frac{1}{2}}} \|w_\varepsilon^3(x_3)\|_{\dot{H}_h^{\frac{1}{2}}}.$$

In particular, using (3.7), we get

$$\begin{aligned} & \int_0^t |\langle \bar{v}_\varepsilon^h \cdot \nabla^h w_\varepsilon^3, w_\varepsilon^3 \rangle_{\dot{H}_h^{\frac{1}{2}}}(t', 0)| dt' \\ & \lesssim \|\nabla^h \bar{v}_\varepsilon^h(\cdot, 0)\|_{L^2(\mathbb{R}^+, L_h^2)} \|\nabla^h w_\varepsilon^3(\cdot, 0)\|_{L^2(\mathbb{R}^+, \dot{H}_h^{\frac{1}{2}})} \|w_\varepsilon^3(\cdot, 0)\|_{L^\infty(\mathbb{R}^+, \dot{H}_h^{\frac{1}{2}})} \\ & \lesssim \|\bar{v}_{\varepsilon,0}^h(\cdot, 0)\|_{L_h^2} \|\nabla^h w_\varepsilon^3(\cdot, 0)\|_{L^2(\mathbb{R}^+, \dot{H}_h^{\frac{1}{2}})} \|w_\varepsilon^3(\cdot, 0)\|_{L^\infty(\mathbb{R}^+, \dot{H}_h^{\frac{1}{2}})} \end{aligned}$$

Then we infer that

$$\begin{aligned} & \int_0^t |\langle \bar{v}_\varepsilon^h \cdot \nabla^h w_\varepsilon^3, w_\varepsilon^3 \rangle_{\dot{H}_h^{\frac{1}{2}}}(t', \cdot, 0)| dt' \lesssim \|u_{\varepsilon,0}(\cdot, 0)\|_{L_h} \\ & \quad \times \left( \|\nabla^h w_\varepsilon^3(\cdot, 0)\|_{L^2(\mathbb{R}^+, \dot{H}_h^{\frac{1}{2}})}^2 + \|w_\varepsilon^3(\cdot, 0)\|_{L^\infty(\mathbb{R}^+, \dot{H}_h^{\frac{1}{2}})}^2 \right). \end{aligned}$$

Plugging this inequality into (3.8) we obtain that there is a constant  $C$  such that

$$\begin{aligned} & \|w_\varepsilon^3(\cdot, 0)\|_{L^\infty(\mathbb{R}^+, \dot{H}_h^{\frac{1}{2}})}^2 + (1 - C\|u_{\varepsilon,0}(\cdot, 0)\|_{L_h^2}) \|\nabla^h w_\varepsilon^3(\cdot, 0)\|_{L^2(\mathbb{R}^+, \dot{H}_h^{\frac{1}{2}})}^2 \\ & \lesssim \|w_{\varepsilon,0}^3(\cdot, 0)\|_{\dot{H}_h^{\frac{1}{2}}}^2 + \|T_\varepsilon(\cdot, 0)\|_{L^2(\mathbb{R}^+, \dot{H}_h^{-\frac{1}{2}})}^2 \\ & \lesssim \|w_{\varepsilon,0}^3(\cdot, 0)\|_{L_h^2} \|w_{\varepsilon,0}^3(\cdot, 0)\|_{\dot{H}_h^1} + \|T_\varepsilon(\cdot, 0)\|_{L^2(\mathbb{R}^+, \dot{H}_h^{-\frac{1}{2}})}^2. \end{aligned}$$

As  $w_{\varepsilon,0}$  is uniformly bounded in  $L_v^\infty \dot{H}_h^1$ , it remains to prove that

$$\lim_{\varepsilon \rightarrow 0} \|T_\varepsilon(\cdot, 0)\|_{L^2(\mathbb{R}^+, \dot{H}_h^{-\frac{1}{2}})} = 0.$$

As  $\partial_3^2 w_\varepsilon^3 = -\partial_3 \text{div}_h w_\varepsilon^h$ , we get

$$\|\partial_3^2 w_\varepsilon^3(\cdot, 0)\|_{L^2(\mathbb{R}^+, \dot{H}_h^{-\frac{1}{2}})} \leq \|\partial_3 \nabla^h w_\varepsilon^h(\cdot, 0)\|_{L^2(\mathbb{R}^+, \dot{H}_h^{-\frac{1}{2}})} \leq \|\partial_3 \nabla^h w_\varepsilon^h\|_{L^2(\mathbb{R}^+, L_v^\infty \dot{H}_h^{-\frac{1}{2}})}.$$

The bounds on  $w_\varepsilon$  given in Proposition 3.3 along with the Gagliardo-Nirenberg inequality lead to

$$\begin{aligned} \|\partial_3^2 w_\varepsilon^3(\cdot, 0)\|_{L^2(\mathbb{R}^+, \dot{H}_h^{-\frac{1}{2}})} & \leq \|\partial_3 \nabla^h w_\varepsilon^h\|_{L^2(\mathbb{R}^+, L_v^2 \dot{H}_h^{-\frac{1}{2}})}^{\frac{1}{2}} \|\partial_3^2 \nabla^h w_\varepsilon^h\|_{L^2(\mathbb{R}^+, L_v^2 \dot{H}_h^{-\frac{1}{2}})}^{\frac{1}{2}} \\ & \lesssim \varepsilon^2. \end{aligned}$$

Now let us turn to the pressure term. Recall that

$$-\Delta q_\varepsilon = \text{div } N_\varepsilon, \quad \text{with} \quad N_\varepsilon \stackrel{\text{def}}{=} \bar{v}_\varepsilon^h \cdot \nabla^h w_\varepsilon = \text{div}_h (\bar{v}_\varepsilon^h \otimes w_\varepsilon)$$

since  $\bar{v}_\varepsilon^h$  is divergence free. To estimate  $\partial_3 q_\varepsilon(\cdot, 0)$  we use Gagliardo-Nirenberg's inequality, according to which it suffices to estimate  $\partial_3 q_\varepsilon$  in  $L_v^2$  and in  $\dot{H}_v^1$ .

Since  $(-\Delta)^{-1} \operatorname{div}_h \operatorname{div}$  is a zero order Fourier multiplier, we have

$$\|\partial_3 q_\varepsilon\|_{L^2(\mathbb{R}^+, H^{-\frac{1}{2}, 1})} \lesssim \|\partial_3(\bar{v}_\varepsilon^h \otimes w_\varepsilon)\|_{L^2(\mathbb{R}^+, H^{-\frac{1}{2}, 1})}.$$

On the one hand we write

$$\|w_\varepsilon \partial_3 \bar{v}_\varepsilon^h\|_{L^2(\mathbb{R}^+, L_v^2 \dot{H}_h^{-\frac{1}{2}})} \lesssim \|w_\varepsilon\|_{L^2(\mathbb{R}^+, L_v^2 \dot{H}_h^{\frac{1}{2}})} \|\partial_3 \bar{v}_\varepsilon^h\|_{L^\infty(\mathbb{R}^+, L_v^\infty L_h^2)} \lesssim \varepsilon^{\frac{1}{2}}$$

by Propositions 3.2 and 3.3, and similarly

$$\|\bar{v}_\varepsilon^h \partial_3 w_\varepsilon\|_{L^2(\mathbb{R}^+, L_v^2 \dot{H}_h^{-\frac{1}{2}})} \lesssim \|\partial_3 w_\varepsilon\|_{L^2(\mathbb{R}^+, L_v^2 \dot{H}_h^{\frac{1}{2}})} \|\bar{v}_\varepsilon^h\|_{L^\infty(\mathbb{R}^+, L_v^\infty L_h^2)} \lesssim \varepsilon^{\frac{1}{2}}.$$

In the same way we find that

$$\|\partial_3(\bar{v}_\varepsilon^h \otimes w_\varepsilon)\|_{L^2(\mathbb{R}^+, \dot{H}^{-\frac{1}{2}, 1})} \lesssim \varepsilon^{\frac{3}{2}}.$$

This ends the proof of Proposition 3.6.  $\square$

#### 4. ESTIMATES ON THE LINEAR TRANSPORT-DIFFUSION EQUATION

In this appendix we shall prove Proposition 3.3. It turns out to be convenient to rescale  $w_\varepsilon$ . Thus we define the vector field

$$W_\varepsilon(t, x) \stackrel{\text{def}}{=} \left( \frac{w_\varepsilon^h}{\varepsilon}, w_\varepsilon^3 \right)(t, x_h, \varepsilon^{-1} x_3)$$

which satisfies

$$\begin{cases} \partial_t W_\varepsilon + \bar{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon - \Delta_h W_\varepsilon - \varepsilon^2 \partial_3^2 W_\varepsilon &= -(\nabla^h Q_\varepsilon, \varepsilon^2 \partial_3 Q_\varepsilon) \\ \operatorname{div} W_\varepsilon &= 0 \\ W_\varepsilon(0, \cdot) &= W_{\varepsilon, 0} \end{cases}$$

where

$$\bar{V}_\varepsilon^h(t, x) \stackrel{\text{def}}{=} \bar{v}_\varepsilon^h(t, x_h, \varepsilon^{-1} x_3) \quad \text{and} \quad Q_\varepsilon(t, x) \stackrel{\text{def}}{=} \varepsilon^{-1} q_\varepsilon(t, x_h, \varepsilon^{-1} x_3).$$

Note that thanks to Proposition 3.2, the vector field  $\partial^\alpha \bar{V}_\varepsilon^h$  is uniformly bounded in the space  $L^\infty(\mathbb{R}^+, L_v^2 \dot{H}_h^s) \cap L^2(\mathbb{R}^+, L_v^2 \dot{H}_h^{s+1})$  for each  $\alpha \in \mathbb{N}^3$  and any  $s > -1$ , and hence also in  $L^\infty(\mathbb{R}^+, L_v^\infty \dot{H}_h^s) \cap L^2(\mathbb{R}^+, L_v^\infty \dot{H}_h^{s+1})$ .

Similary we have defined

$$W_{\varepsilon, 0}(x) \stackrel{\text{def}}{=} \left( \frac{w_{\varepsilon, 0}^h}{\varepsilon}, w_{\varepsilon, 0}^3 \right)(x_h, \varepsilon^{-1} x_3)$$

and by construction it is bounded in  $\dot{H}^s(\mathbb{R}^3)$  for all  $s \geq -1$ .

Proposition 3.3 is a corollary of the next statement.

**Proposition 4.1.** *Under the assumptions of Theorem 3, the following results hold.*

- (1) *For all  $s > -1$ , and all  $\alpha \in \mathbb{N}^3$ ,  $\partial^\alpha W_\varepsilon$  is bounded in  $L^\infty(\mathbb{R}^+, L_v^2 \dot{H}_h^s) \cap L^2(\mathbb{R}^+, L_v^2 \dot{H}_h^{s+1})$ ; in particular  $\partial^\alpha W_\varepsilon$  is bounded in  $L^\infty(\mathbb{R}^+, L_v^\infty \dot{H}_h^s) \cap L^2(\mathbb{R}^+, L_v^\infty \dot{H}_h^{s+1})$ .*
- (2) *For all  $\alpha \in \mathbb{N}^3$ ,  $\partial^\alpha W_\varepsilon$  is bounded in  $L^2(\mathbb{R}^+, L^2)$ , hence in particular in  $L^2(\mathbb{R}^+, L_v^\infty L_h^2)$ .*

*Proof.* Let us start by proving the first statement of the proposition. We notice that it is enough to prove the result for  $s \in ]-1, 1[$ , and we shall argue by induction on  $\alpha$ .

• Let us start by considering the case  $\alpha = 0$ . An energy estimate in  $L_v^2 \dot{H}_h^s$  on the equation satisfied by  $W_\varepsilon$  gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 + \|\nabla^h W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 + \varepsilon^2 \|\partial_3 W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 \\ = -\langle \bar{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon, W_\varepsilon \rangle_{L_v^2 \dot{H}_h^s} - \langle \nabla^h Q_\varepsilon, W_\varepsilon^h \rangle_{L_v^2 \dot{H}_h^s} - \langle \varepsilon^2 \partial_3 Q_\varepsilon, W_\varepsilon^3 \rangle_{L_v^2 \dot{H}_h^s}. \end{aligned}$$

For the non-linear term we have, by [4, Lemma 1.1] and for each given  $t$  and  $x_3$ ,

$$\begin{aligned} |\langle \bar{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon, W_\varepsilon \rangle_{\dot{H}_h^s}(t, x_3)| &\lesssim \|\nabla^h \bar{V}_\varepsilon^h(t, x_3)\|_{L_h^2} \|\nabla^h W_\varepsilon(t, x_3)\|_{\dot{H}_h^s} \|W_\varepsilon(t, x_3)\|_{\dot{H}_h^s} \\ &\leq \frac{1}{4} \|\nabla^h W_\varepsilon(t, x_3)\|_{\dot{H}_h^s}^2 + C \|\nabla^h \bar{V}_\varepsilon^h(t, x_3)\|_{L_h^2}^2 \|W_\varepsilon(t, x_3)\|_{\dot{H}_h^s}^2 \end{aligned}$$

so after integration over  $x_3$ , we find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 + \frac{3}{4} \|\nabla^h W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 + \varepsilon^2 \|\partial_3 W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 \\ \leq C \|\nabla^h \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^2 \|W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 - \langle \nabla^h Q_\varepsilon, W_\varepsilon^h \rangle_{L_v^2 \dot{H}_h^s} - \langle \varepsilon^2 \partial_3 Q_\varepsilon, W_\varepsilon^3 \rangle_{L_v^2 \dot{H}_h^s}. \end{aligned}$$

Now let us study the pressure term. As  $W_\varepsilon$  is a divergence free vector field we have

$$-\langle \nabla^h Q_\varepsilon, W_\varepsilon^h \rangle_{L_v^2 \dot{H}_h^s} - \langle \varepsilon^2 \partial_3 Q_\varepsilon, W_\varepsilon^3 \rangle_{L_v^2 \dot{H}_h^s} = (\varepsilon^2 - 1) \langle \nabla^h Q_\varepsilon, W_\varepsilon^h \rangle_{L_v^2 \dot{H}_h^s}.$$

We claim that

$$(4.1) \quad |\langle \nabla^h Q_\varepsilon(t), W_\varepsilon^h(t) \rangle_{L_v^2 \dot{H}_h^s}| \leq \frac{1}{4} \|\nabla^h W_\varepsilon(t)\|_{L_v^2 \dot{H}_h^s}^2 + C_\varepsilon(t) \|W_\varepsilon(t)\|_{L_v^2 \dot{H}_h^s}^2$$

where  $C_\varepsilon$  is uniformly bounded in  $L^1(\mathbb{R}^+)$ . Assuming that claim to be true, we infer (up to changing  $C_\varepsilon$ ) that

$$\frac{d}{dt} \|W_\varepsilon(t)\|_{L_v^2 \dot{H}_h^s}^2 + \|\nabla^h W_\varepsilon(t)\|_{L_v^2 \dot{H}_h^s}^2 + \varepsilon^2 \|\partial_3 W_\varepsilon(t)\|_{L_v^2 \dot{H}_h^s}^2 \lesssim C_\varepsilon(t) \|W_\varepsilon(t)\|_{L_v^2 \dot{H}_h^s}^2.$$

Thanks to Gronwall's lemma this gives

$$\|W_\varepsilon(t)\|_{L_v^2 \dot{H}_h^s}^2 + \int_0^t \|\nabla^h W_\varepsilon(t')\|_{L_v^2 \dot{H}_h^s}^2 dt' \lesssim \|W_{\varepsilon,0}\|_{L_v^2 \dot{H}_h^s}^2,$$

and the conclusion of Proposition 4.1 (1), for  $\alpha = 0$  and  $-1 < s < 1$ , comes from the a priori bounds on  $W_{\varepsilon,0}$ . It remains to prove the claim (4.1). For all real numbers  $r$ , we have

$$|\langle \nabla^h Q_\varepsilon(t), W_\varepsilon^h(t) \rangle_{L_v^2 \dot{H}_h^s}| \leq \|\nabla^h Q_\varepsilon(t)\|_{L_v^2 \dot{H}_h^r} \|W_\varepsilon^h(t)\|_{L_v^2 \dot{H}_h^{2s-r}}.$$

As  $W_\varepsilon$  is a divergence free vector field we can write

$$\operatorname{div}(\bar{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon) = -\Delta_h Q_\varepsilon - \varepsilon^2 \partial_3^2 Q_\varepsilon.$$

Then we define

$$M_\varepsilon^h \stackrel{\text{def}}{=} \bar{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon^h + \partial_3(W_\varepsilon^3 \bar{V}_\varepsilon^h)$$

and using the fact that  $\bar{V}_\varepsilon^h$  is divergence free, we have

$$\operatorname{div}(\bar{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon) = \operatorname{div}_h M_\varepsilon^h.$$

It follows that

$$(4.2) \quad Q_\varepsilon = (-\Delta_h - \varepsilon^2 \partial_3^2)^{-1} \operatorname{div}_h M_\varepsilon^h,$$

and since  $\nabla^h(-\Delta_h - \varepsilon^2 \partial_3^2)^{-1} \operatorname{div}_h$  is a zero-order Fourier multiplier, we infer that for all real numbers  $r$ ,

$$\|\nabla^h Q_\varepsilon\|_{L_v^2 \dot{H}_h^r} \leq \|M_\varepsilon^h\|_{L_v^2 \dot{H}_h^r},$$

and therefore

$$(4.3) \quad |\langle \nabla^h Q_\varepsilon(t), W_\varepsilon^h(t) \rangle_{L_v^2 \dot{H}_h^s}| \leq \|M_\varepsilon^h(t)\|_{L_v^2 \dot{H}_h^r} \|W_\varepsilon^h(t)\|_{L_v^2 \dot{H}_h^{2s-r}}.$$

We can estimate  $\|M_\varepsilon^h\|_{L_v^2 \dot{H}_h^r}$  as follows, thanks to the divergence-free condition on  $W_\varepsilon$ :

$$\begin{aligned} \|M_\varepsilon^h\|_{L_v^2 \dot{H}_h^r} &\leq \|\bar{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon\|_{L_v^2 \dot{H}_h^r} + \|\partial_3(W_\varepsilon^3 \bar{V}_\varepsilon^h)\|_{L_v^2 \dot{H}_h^r} \\ &\leq \|\bar{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon\|_{L_v^2 \dot{H}_h^r} + \|W_\varepsilon^3 \partial_3 \bar{V}_\varepsilon^h\|_{L_v^2 \dot{H}_h^r} + \|\bar{V}_\varepsilon^h \operatorname{div}_h W_\varepsilon^h\|_{L_v^2 \dot{H}_h^r}. \end{aligned}$$

Thanks to two-dimensional product laws, if  $-1 < r < 0$  then we get

$$\|\bar{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon\|_{L_v^2 \dot{H}_h^r} + \|\bar{V}_\varepsilon^h \operatorname{div}_h W_\varepsilon^h\|_{L_v^2 \dot{H}_h^r} \lesssim \|\bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}} \|\nabla^h W_\varepsilon\|_{L_v^2 \dot{H}_h^{r+\frac{1}{2}}}$$

and

$$\|W_\varepsilon^3 \partial_3 \bar{V}_\varepsilon^h\|_{L_v^2 \dot{H}_h^r} \lesssim \|\nabla \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2} \|W_\varepsilon^3\|_{L_v^2 \dot{H}_h^{r+1}}.$$

So if  $-1 < r < 0$ , then

$$(4.4) \quad \|M_\varepsilon^h\|_{L_v^2 \dot{H}_h^r} \lesssim \|\bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}} \|\nabla^h W_\varepsilon\|_{L_v^2 \dot{H}_h^{r+\frac{1}{2}}} + \|\nabla \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2} \|W_\varepsilon^3\|_{L_v^2 \dot{H}_h^{r+1}}$$

and this leads to (4.1) for  $-1 < s < 1$ , due to the following computations.

◦ If  $0 < s < 1$ , we choose  $r = s - 1$  to get

$$\|M_\varepsilon^h\|_{L_v^2 \dot{H}_h^{s-1}} \lesssim \|\bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}} \|\nabla^h W_\varepsilon\|_{L_v^2 \dot{H}_h^{s-\frac{1}{2}}} + \|\nabla \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2} \|W_\varepsilon^3\|_{L_v^2 \dot{H}_h^s},$$

so by (4.3) with  $r = s - 1$ , we infer that

$$(4.5) \quad \begin{aligned} |\langle \nabla^h Q_\varepsilon, W_\varepsilon^h \rangle_{L_v^2 \dot{H}_h^s}| &\lesssim \|\bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}} \|\nabla^h W_\varepsilon\|_{L_v^2 \dot{H}_h^{s-\frac{1}{2}}} \|\nabla^h W_\varepsilon\|_{L_v^2 \dot{H}_h^s} \\ &\quad + \frac{1}{8} \|\nabla^h W_\varepsilon^h\|_{L_v^2 \dot{H}_h^s}^2 + C \|\nabla \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^2 \|W_\varepsilon^3\|_{L_v^2 \dot{H}_h^s}^2. \end{aligned}$$

We then use the interpolation inequality

$$\|\bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}} \|\nabla^h W_\varepsilon\|_{L_v^2 \dot{H}_h^{s-\frac{1}{2}}} \lesssim \|\bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^{\frac{1}{2}} \|\nabla^h \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^{\frac{1}{2}} \|W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^{\frac{1}{2}} \|\nabla^h W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^{\frac{1}{2}}$$

along with the convexity inequality  $ab \leq \frac{3}{4}a^{4/3} + \frac{1}{4}b^4$ , to get

$$\begin{aligned} \|\nabla^h W_\varepsilon\|_{L_v^2 \dot{H}_h^s} \|\bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}} \|\nabla^h W_\varepsilon\|_{L_v^2 \dot{H}_h^{s-\frac{1}{2}}} &\leq \frac{1}{8} \|\nabla^h W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 \\ &\quad + C \|\bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^2 \|\nabla \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^2 \|W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2. \end{aligned}$$

It remains to define

$$(4.6) \quad C_\varepsilon(t) \stackrel{\text{def}}{=} C \|\nabla \bar{V}_\varepsilon^h(t)\|_{L_v^\infty L_h^2}^2 (1 + \|\bar{V}_\varepsilon^h(t)\|_{L_v^\infty L_h^2}^2)$$

to obtain from (4.5) that

$$|\langle \nabla^h Q_\varepsilon(t), W_\varepsilon^h(t) \rangle_{L_v^2 \dot{H}_h^s}| \leq \frac{1}{4} \|\nabla^h W_\varepsilon(t)\|_{L_v^2 \dot{H}_h^s}^2 + C_\varepsilon(t) \|W_\varepsilon(t)\|_{L_v^2 \dot{H}_h^s}^2.$$

Notice that  $C_\varepsilon$  belongs to  $L^1(\mathbb{R}^+)$  thanks to the uniform bounds on  $\bar{V}_\varepsilon^h$  derived above from Proposition 3.2.

◦ If  $s = 0$ , we choose  $r = -\frac{1}{2}$  and hence by (4.3) and (4.4),

$$|\langle \nabla^h Q_\varepsilon, W_\varepsilon^h \rangle_{L^2}| \lesssim \|W_\varepsilon^h\|_{L_v^2 \dot{H}_h^{\frac{1}{2}}} (\|\bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}} \|\nabla^h W_\varepsilon\|_{L^2} + \|\nabla \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2} \|W_\varepsilon^3\|_{L_v^2 \dot{H}_h^{\frac{1}{2}}}).$$

By interpolation we infer that

$$\begin{aligned} |\langle \nabla^h Q_\varepsilon, W_\varepsilon^h \rangle_{L^2}| &\lesssim \|W_\varepsilon^h\|_{L^2}^{\frac{1}{2}} \|\nabla^h W_\varepsilon^h\|_{L^2}^{\frac{3}{2}} \|\bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^{\frac{1}{2}} \|\nabla \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^{\frac{1}{2}} \\ &\quad + \|W_\varepsilon\|_{L^2} \|\nabla^h W_\varepsilon\|_{L^2} \|\nabla \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}. \end{aligned}$$

The convexity inequality  $ab \leq \frac{3}{4}a^{4/3} + \frac{1}{4}b^4$  implies that

$$\begin{aligned} (4.7) \quad &\|W_\varepsilon^h\|_{L^2}^{\frac{1}{2}} \|\nabla^h W_\varepsilon^h\|_{L^2}^{\frac{3}{2}} \|\bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^{\frac{1}{2}} \|\nabla \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^{\frac{1}{2}} \\ &\leq \frac{1}{8} \|\nabla^h W_\varepsilon\|_{L^2}^2 + C \|W_\varepsilon^h\|_{L^2}^2 \|\bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^2 \|\nabla \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^2 \end{aligned}$$

and

$$(4.8) \quad \|W_\varepsilon\|_{L^2} \|\nabla^h W_\varepsilon\|_{L^2} \|\nabla \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2} \leq \frac{1}{8} \|\nabla^h W_\varepsilon\|_{L^2}^2 + C \|W_\varepsilon\|_{L^2}^2 \|\nabla \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^2.$$

With the above choice (4.6) for  $C_\varepsilon$  we obtain

$$|\langle \nabla^h Q_\varepsilon(t), W_\varepsilon^h(t) \rangle_{L^2}| \leq \frac{1}{4} \|\nabla^h W_\varepsilon(t)\|_{L^2}^2 + C_\varepsilon(t) \|W_\varepsilon(t)\|_{L^2}^2.$$

◦ Finally if  $-1 < s < 0$ , we proceed slightly differently. We recall that

$$\operatorname{div}_h M_\varepsilon^h = -\Delta_h Q_\varepsilon - \varepsilon^2 \partial_3^2 Q_\varepsilon$$

and as  $W_\varepsilon$  is divergence free, we have

$$M_\varepsilon^h = \bar{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon^h - \bar{V}_\varepsilon^h \operatorname{div}_h W_\varepsilon^h + W_\varepsilon^3 \partial_3 \bar{V}_\varepsilon^h.$$

Defining

$$M_{\varepsilon,1}^h \stackrel{\text{def}}{=} \operatorname{div}_h (\bar{V}_\varepsilon^h \otimes W_\varepsilon^h - W_\varepsilon^h \otimes \bar{V}_\varepsilon^h) \quad \text{and} \quad M_{\varepsilon,2}^h \stackrel{\text{def}}{=} W_\varepsilon \cdot \nabla \bar{V}_\varepsilon^h,$$

we can split  $M_\varepsilon^h = M_{\varepsilon,1}^h + M_{\varepsilon,2}^h$  and estimate each term differently.

Since  $\nabla^h(-\Delta_h - \varepsilon^2 \partial_3^2) \operatorname{div}_h$  is a zero-order Fourier multiplier,

$$|\langle \nabla^h Q_\varepsilon, W_\varepsilon^h \rangle_{L_v^2 \dot{H}_h^s}| \leq \|M_{\varepsilon,1}^h\|_{L_v^2 \dot{H}_h^{s-1}} \|W_\varepsilon^h\|_{L_v^2 \dot{H}_h^{s+1}} + \|M_{\varepsilon,2}^h\|_{L_v^2 \dot{H}_h^s} \|W_\varepsilon^h\|_{L_v^2 \dot{H}_h^s}.$$

Using two-dimensional product laws we obtain

$$\|M_{\varepsilon,1}^h\|_{L_v^2 \dot{H}_h^{s-1}} \lesssim \|\bar{V}_\varepsilon^h W_\varepsilon\|_{L_v^2 \dot{H}_h^s} \lesssim \|\bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}} \|W_\varepsilon\|_{L_v^2 \dot{H}_h^{s+\frac{1}{2}}}$$

and

$$\|M_{\varepsilon,2}^h\|_{L_v^2 \dot{H}_h^s} \lesssim \|W_\varepsilon \cdot \nabla \bar{V}_\varepsilon^h\|_{L_v^2 \dot{H}_h^s} \lesssim \|\nabla \bar{V}_\varepsilon^h\|_{L_v^\infty L^2} \|W_\varepsilon\|_{L_v^2 \dot{H}_h^{s+1}}.$$



Therefore, we get

$$(4.9) \quad \begin{aligned} |\langle \nabla^h Q_\varepsilon, W_\varepsilon^h \rangle_{L_v^2 \dot{H}_h^s}| &\leq \|\bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}} \|W_\varepsilon\|_{L_v^2 \dot{H}_h^{s+\frac{1}{2}}} \|\nabla^h W_\varepsilon^h\|_{L_v^2 \dot{H}_h^s} \\ &\quad + \|\nabla \bar{V}_\varepsilon^h\|_{L_v^\infty L^2} \|\nabla^h W_\varepsilon\|_{L_v^2 \dot{H}_h^s} \|W_\varepsilon^h\|_{L_v^2 \dot{H}_h^s}. \end{aligned}$$

Then we use the interpolation inequality

$$\|W_\varepsilon\|_{L_v^2 \dot{H}_h^{s+\frac{1}{2}}} \|\nabla^h W_\varepsilon^h\|_{L_v^2 \dot{H}_h^s} \lesssim \|W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^{\frac{1}{2}} \|\nabla^h W_\varepsilon^h\|_{L_v^2 \dot{H}_h^s}^{3/2}$$

along with the convexity inequalities  $ab \leq \frac{3}{4}a^{4/3} + \frac{1}{4}b^4$  and  $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ , to infer that again with the choice (4.6) for  $C_\varepsilon$ ,

$$|\langle \nabla^h Q_\varepsilon(t), W_\varepsilon^h(t) \rangle_{L_v^2 \dot{H}_h^s}| \leq \frac{1}{4} \|\nabla^h W_\varepsilon(t)\|_{L_v^2 \dot{H}_h^s}^2 + C_\varepsilon(t) \|W_\varepsilon(t)\|_{L_v^2 \dot{H}_h^s}^2.$$

The first result of the proposition is therefore proved in the case when  $\alpha = 0$  and  $-1 < s < 1$ .

• To go further in the induction process, let  $k \in \mathbb{N}$  be given and suppose the result proved for all  $\alpha \in \mathbb{N}^3$  such that  $|\alpha| \leq k$ , still for  $-1 < s < 1$ . Now consider  $\alpha \in \mathbb{N}^3$  such that  $|\alpha| = k + 1$ . The vector field  $\partial^\alpha W_\varepsilon$  solves

$$\partial_t \partial^\alpha W_\varepsilon + \partial^\alpha (\bar{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon) - \Delta_h \partial^\alpha W_\varepsilon - \varepsilon^2 \partial_3^2 \partial^\alpha W_\varepsilon = -(\nabla^h \partial^\alpha Q_\varepsilon, \varepsilon^2 \partial_3 \partial^\alpha Q_\varepsilon).$$

An energy estimate in  $L_v^2 \dot{H}_h^s$  gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial^\alpha W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 + \langle \partial^\alpha (\bar{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon), \partial^\alpha W_\varepsilon \rangle_{L_v^2 \dot{H}_h^s} + \|\nabla^h \partial^\alpha W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 \\ = -\langle \nabla^h \partial^\alpha Q_\varepsilon, \partial^\alpha W_\varepsilon^h \rangle_{L_v^2 \dot{H}_h^s} - \varepsilon^2 \langle \partial_3 \partial^\alpha Q_\varepsilon, \partial^\alpha W_\varepsilon^3 \rangle_{L_v^2 \dot{H}_h^s}. \end{aligned}$$

We split  $\langle \partial^\alpha (\bar{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon), \partial^\alpha W_\varepsilon \rangle_{L_v^2 \dot{H}_h^s}$  into two contributions:

$$(4.10) \quad \langle \bar{V}_\varepsilon^h \cdot \nabla^h \partial^\alpha W_\varepsilon, \partial^\alpha W_\varepsilon \rangle_{L_v^2 \dot{H}_h^s} + \sum_{0 < \beta \leq \alpha} C_\beta \langle \partial^\beta \bar{V}_\varepsilon^h \cdot \nabla^h \partial^{\alpha-\beta} W_\varepsilon, \partial^\alpha W_\varepsilon \rangle_{L_v^2 \dot{H}_h^s}.$$

The first term in (4.10) satisfies, as in [4, Lemma 1.1]

$$\begin{aligned} |\langle \bar{V}_\varepsilon^h \cdot \nabla^h \partial^\alpha W_\varepsilon, \partial^\alpha W_\varepsilon \rangle_{\dot{H}_h^s}| &\lesssim \|\nabla^h \bar{V}_\varepsilon^h\|_{L_h^2} \|\nabla^h \partial^\alpha W_\varepsilon\|_{\dot{H}_h^s} \|\partial^\alpha W_\varepsilon\|_{\dot{H}_h^s} \\ &\leq \frac{1}{4} \|\nabla^h \partial^\alpha W_\varepsilon\|_{\dot{H}_h^s}^2 + C \|\nabla^h \bar{V}_\varepsilon^h\|_{L_h^2}^2 \|\partial^\alpha W_\varepsilon\|_{\dot{H}_h^s}^2 \end{aligned}$$

so

$$|\langle \bar{V}_\varepsilon^h \cdot \nabla^h \partial^\alpha W_\varepsilon, \partial^\alpha W_\varepsilon \rangle_{L_v^2 \dot{H}_h^s}| \leq \frac{1}{4} \|\nabla^h \partial^\alpha W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 + C \|\nabla^h \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^2 \|\partial^\alpha W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2.$$

For the remaining terms in (4.10), as  $\bar{V}_\varepsilon^h$  is a horizontal, divergence free vector field, two-dimensional product laws give

$$\begin{aligned} |\langle \partial^\beta \bar{V}_\varepsilon^h \cdot \nabla^h \partial^{\alpha-\beta} W_\varepsilon, \partial^\alpha W_\varepsilon \rangle_{\dot{H}_h^s}| &= |\langle \operatorname{div}_h (\partial^\beta \bar{V}_\varepsilon^h \otimes \partial^{\alpha-\beta} W_\varepsilon), \partial^\alpha W_\varepsilon \rangle_{\dot{H}_h^s}| \\ &\lesssim \|\partial^\beta \bar{V}_\varepsilon^h \otimes \partial^{\alpha-\beta} W_\varepsilon\|_{\dot{H}_h^s} \|\nabla^h \partial^\alpha W_\varepsilon\|_{\dot{H}_h^s} \\ &\lesssim \|\partial^\beta \bar{V}_\varepsilon^h\|_{\dot{H}_h^{\frac{s+1}{2}}} \|\partial^{\alpha-\beta} W_\varepsilon\|_{\dot{H}_h^{\frac{s+1}{2}}} \|\nabla^h \partial^\alpha W_\varepsilon\|_{\dot{H}_h^s} \end{aligned}$$

so

$$|\langle \partial^\beta \bar{V}_\varepsilon^h \cdot \nabla^h \partial^{\alpha-\beta} W_\varepsilon, \partial^\alpha W_\varepsilon \rangle_{L_v^2 \dot{H}_h^s}| \leq \frac{1}{4} \|\nabla^h \partial^\alpha W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 + C \|\partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{s+1}{2}}}^2 \|\partial^{\alpha-\beta} W_\varepsilon\|_{L_v^2 \dot{H}_h^{\frac{s+1}{2}}}^2.$$

Then we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial^\alpha W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 + \frac{1}{2} \|\nabla^h \partial^\alpha W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 + \varepsilon^2 \|\partial_3 \partial^\alpha W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 &\lesssim \|\nabla^h \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^2 \|\partial^\alpha W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 \\ &+ |\langle \nabla^h \partial^\alpha Q_\varepsilon, \partial^\alpha W_\varepsilon^h \rangle_{L_v^2 \dot{H}_h^s} - \varepsilon^2 \langle \partial_3 \partial^\alpha Q_\varepsilon, \partial^\alpha W_\varepsilon^3 \rangle_{L_v^2 \dot{H}_h^s}| \\ &+ C \sum_{0 < \beta \leq \alpha} \|\partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{s+1}{2}}}^2 \|\partial^{\alpha-\beta} W_\varepsilon\|_{L_v^2 \dot{H}_h^{\frac{s+1}{2}}}^2. \end{aligned}$$

Now let us estimate the pressure term. We recall that

$$-\langle \nabla^h \partial^\alpha Q_\varepsilon, \partial^\alpha W_\varepsilon^h \rangle_{L_v^2 \dot{H}_h^s} - \langle \varepsilon^2 \partial_3 \partial^\alpha Q_\varepsilon, \partial^\alpha W_\varepsilon^3 \rangle_{L_v^2 \dot{H}_h^s} = (\varepsilon^2 - 1) \langle \nabla^h \partial^\alpha Q_\varepsilon, \partial^\alpha W_\varepsilon^h \rangle_{L_v^2 \dot{H}_h^s}$$

and we claim that

$$(4.11) \quad |\langle \nabla^h \partial^\alpha Q_\varepsilon, \partial^\alpha W_\varepsilon^h \rangle_{L_v^2 \dot{H}_h^s}(t)| \leq \frac{1}{4} \|\nabla^h \partial^\alpha W_\varepsilon(t)\|_{L_v^2 \dot{H}_h^s}^2 + C_{1,\varepsilon}(t) + C_{2,\varepsilon}(t) \|\partial^\alpha W_\varepsilon(t)\|_{L_v^2 \dot{H}_h^s}^2$$

with  $C_{1,\varepsilon}$  and  $C_{2,\varepsilon}$  uniformly bounded in  $L_1(\mathbb{R}^+)$ . By the induction assumption (noticing that  $(s+1)/2 + \alpha - 1 < \alpha$ ) we deduce that  $\sum_{0 < \beta \leq \alpha} \|\partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{s+1}{2}}}^2 \|\partial^{\alpha-\beta} W_\varepsilon\|_{L_v^2 \dot{H}_h^{\frac{s+1}{2}}}^2$  is uniformly

bounded in  $L^1(\mathbb{R}^+)$  so up to changing  $C_{1,\varepsilon}$  and  $C_{2,\varepsilon}$  we get

$$\frac{d}{dt} \|\partial^\alpha W_\varepsilon(t)\|_{L_v^2 \dot{H}_h^s}^2 + \|\nabla^h \partial^\alpha W_\varepsilon(t)\|_{L_v^2 \dot{H}_h^s}^2 \leq C_{1,\varepsilon}(t) + C_{2,\varepsilon}(t) \|\partial^\alpha W_\varepsilon(t)\|_{L_v^2 \dot{H}_h^s}^2.$$

Using Gronwall's lemma in turn this implies that

$$\|\partial^\alpha W_\varepsilon(t)\|_{L_v^2 \dot{H}_h^s}^2 + \int_0^t \|\nabla^h \partial^\alpha W_\varepsilon(t')\|_{L_v^2 \dot{H}_h^s}^2 dt' \lesssim \|\partial^\alpha W_{\varepsilon,0}\|_{L_v^2 \dot{H}_h^s}^2$$

and the bounds on  $W_{\varepsilon,0}$  conclude the proof if  $-1 < s < 1$ . It remains to prove the estimate (4.11) on the pressure term. We shall adapt the computations of the case  $\alpha = 0$ . We define

$$N_{\varepsilon,\alpha,\beta} \stackrel{\text{def}}{=} \partial^\beta \bar{V}_\varepsilon^h \cdot \nabla^h \partial^{\alpha-\beta} W_\varepsilon^h + \partial_3(\partial^{\alpha-\beta} W_\varepsilon^3 \partial^\beta \bar{V}_\varepsilon^h)$$

and recalling (4.2) we get, since  $\nabla^h(-\Delta_h - \varepsilon^2 \partial_3^2)^{-1} \text{div}_h$  is a Fourier multiplier of order 0,

$$\langle \nabla^h \partial^\alpha Q_\varepsilon, \partial^\alpha W_\varepsilon^h \rangle_{L_v^2 \dot{H}_h^s} \lesssim \sum_{0 \leq \beta \leq \alpha} \|N_{\varepsilon,\alpha,\beta}\|_{L_v^2 \dot{H}_h^{r_\beta}} \|\partial^\alpha W_\varepsilon^h(t, \cdot)\|_{L_v^2 \dot{H}_h^{2s-r_\beta}}$$

where  $r_\beta$  is any real number. Then we define

$$(*)_{\alpha,\beta} := \|N_{\varepsilon,\alpha,\beta}\|_{L_v^2 \dot{H}_h^{r_\beta}} \|\partial^\alpha W_\varepsilon^h(t, \cdot)\|_{L_v^2 \dot{H}_h^{2s-r_\beta}}.$$

The term  $(*)_{\alpha,0}$  can be treated as was done for  $\alpha = 0$ , changing  $W_\varepsilon^h$  into  $\partial^\alpha W_\varepsilon^h$ . So we have, as in the proof of (4.1),

$$(4.12) \quad |(*)_{\alpha,0}| \leq \frac{1}{8} \|\nabla^h \partial^\alpha W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 + C \|\partial^\alpha W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 \|\nabla^h \partial^\alpha \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^2 (1 + \|\partial^\alpha \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^2).$$

For the others terms we have the following estimates.

◦ If  $0 < s < 1$  we choose  $r_\beta = s - 1$  like in the case  $\alpha = 0$ , and as in (4.5) we obtain

$$\begin{aligned} \sum_{0 < \beta \leq \alpha} |(*)_{\alpha, \beta}| &\leq \frac{1}{8} \|\nabla^h \partial^\alpha W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 + C \sum_{0 < \beta \leq \alpha} \|\partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}}^2 \|\nabla^h \partial^{\alpha-\beta} W_\varepsilon\|_{L_v^2 \dot{H}_h^{s-\frac{1}{2}}}^2 \\ &\quad + C \sum_{0 < \beta \leq \alpha} \|\nabla \partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^2 \|\partial^{\alpha-\beta} W_\varepsilon^3\|_{L_v^2 \dot{H}_h^s}^2. \end{aligned}$$

Then we define, recalling (4.12),

$$C_{1,\varepsilon} \stackrel{\text{def}}{=} C \sum_{0 < \beta \leq \alpha} \|\partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}}^2 \|\nabla^h \partial^{\alpha-\beta} W_\varepsilon\|_{L_v^2 \dot{H}_h^{s-\frac{1}{2}}}^2 + C \sum_{0 < \beta \leq \alpha} \|\nabla \partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^2 \|\partial^{\alpha-\beta} W_\varepsilon^3\|_{L_v^2 \dot{H}_h^s}^2$$

and

$$C_{2,\varepsilon} \stackrel{\text{def}}{=} C \|\nabla^h \partial^\alpha \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^2 (1 + \|\partial^\alpha \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^2)$$

to get

$$\sum_{0 \leq \beta \leq \alpha} |(*)_{\alpha, \beta}| \leq \frac{1}{4} \|\nabla^h \partial^\alpha W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 + C_{1,\varepsilon} + C_{2,\varepsilon} \|\partial^\alpha W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2.$$

Note that the families  $(C_{1,\varepsilon})_{\varepsilon>0}$  and  $(C_{2,\varepsilon})_{\varepsilon>0}$  are bounded in  $L^1(\mathbb{R}^+)$  thanks to the induction assumption and Proposition 3.2.

◦ If  $s = 0$  then following the steps leading to (4.7)-(4.8) we choose  $r_\beta = -1/2$  and write

$$|(*)_{\alpha, \beta}| \lesssim \|\partial^\alpha W_\varepsilon^h\|_{L_v^2 \dot{H}_h^{\frac{1}{2}}} \left( \|\partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}} \|\nabla^h \partial^{\alpha-\beta} W_\varepsilon\|_{L^2} + \|\nabla \partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2} \|\partial^{\alpha-\beta} W_\varepsilon^3\|_{L_v^2 \dot{H}_h^{\frac{1}{2}}} \right)$$

so, by interpolation, we get

$$\begin{aligned} |(*)_{\alpha, \beta}| &\lesssim \|\partial^\alpha W_\varepsilon^h\|_{L^2}^{\frac{1}{2}} \|\partial^\alpha \nabla^h W_\varepsilon^h\|_{L^2}^{\frac{1}{2}} \|\partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}} \|\nabla^h \partial^{\alpha-\beta} W_\varepsilon\|_{L^2} \\ &\quad + \|\partial^\alpha W_\varepsilon^h\|_{L^2}^{\frac{1}{2}} \|\partial^\alpha \nabla^h W_\varepsilon^h\|_{L^2}^{\frac{1}{2}} \|\nabla \partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2} \|\partial^{\alpha-\beta} W_\varepsilon^3\|_{L_v^2 \dot{H}_h^{\frac{1}{2}}}. \end{aligned}$$

When  $\beta > 0$ , the convexity inequality  $abc \leq \frac{1}{4}a^4 + \frac{1}{4}b^4 + \frac{1}{2}c^2$  leads to

$$\begin{aligned} \sum_{0 < \beta \leq \alpha} |(*)_{\alpha, \beta}| &\leq \frac{1}{8} \|\nabla^h \partial^\alpha W_\varepsilon\|_{L^2}^2 + C \sum_{0 < \beta \leq \alpha} \|\partial^\alpha W_\varepsilon^h\|_{L^2}^2 (\|\partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}}^4 + \|\nabla \partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^4) \\ &\quad + C \sum_{0 < \beta \leq \alpha} (\|\nabla^h \partial^{\alpha-\beta} W_\varepsilon\|_{L^2}^2 + \|\nabla^h \partial^{\alpha-\beta} W_\varepsilon^3\|_{L_v^2 \dot{H}_h^{-\frac{1}{2}}}^2). \end{aligned}$$

We define

$$C_{1,\varepsilon} \stackrel{\text{def}}{=} C \sum_{0 < \beta \leq \alpha} (\|\nabla^h \partial^{\alpha-\beta} W_\varepsilon\|_{L^2}^2 + \|\nabla^h \partial^{\alpha-\beta} W_\varepsilon^3\|_{L_v^2 \dot{H}_h^{-\frac{1}{2}}}^2)$$

and

$$C_{2,\varepsilon} \stackrel{\text{def}}{=} C \|\nabla^h \partial^\alpha \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^2 (1 + \|\partial^\alpha \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^2) + C \sum_{0 < \beta \leq \alpha} (\|\partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}}^4 + \|\nabla \partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty L_h^2}^4)$$

to get when  $s = 0$  and recalling (4.12),

$$\sum_{0 < \beta \leq \alpha} |(*)_{\alpha, \beta}| \leq \frac{1}{4} \|\nabla^h \partial^\alpha W_\varepsilon\|_{L^2}^2 + C_{1,\varepsilon} + C_{2,\varepsilon} \|\partial^\alpha W_\varepsilon\|_{L^2}^2.$$

Again note that the families  $(C_{1,\varepsilon})_{\varepsilon>0}$  and  $(C_{2,\varepsilon})_{\varepsilon>0}$  are bounded in  $L^1(\mathbb{R}^+)$  thanks to the induction assumption and Proposition 3.2.

◦ If  $-1 < s < 0$  then following the computations leading to (4.9), we write

$$\begin{aligned} |(*)_{\alpha,\beta}| &\lesssim \|\partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}} \|\partial^{\alpha-\beta} W_\varepsilon\|_{L_v^2 \dot{H}_h^{s+\frac{1}{2}}} \|\nabla \partial^\alpha W_\varepsilon^h\|_{L_v^2 \dot{H}_h^s} \\ &\quad + \|\nabla \partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty L^2} \|\nabla \partial^{\alpha-\beta} W_\varepsilon\|_{L_v^2 \dot{H}_h^s} \|\partial^\alpha W_\varepsilon^h\|_{L_v^2 \dot{H}_h^s} \end{aligned}$$

so

$$\begin{aligned} \sum_{0 < \beta \leq \alpha} |(*)_{\alpha,\beta}| &\leq \frac{1}{4} \|\nabla^h \partial^\alpha W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 + \sum_{\beta < \alpha} \|\partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}}^2 \|\partial^{\alpha-\beta} W_\varepsilon\|_{L_v^2 \dot{H}_h^{s+\frac{1}{2}}}^2 \\ &\quad + \sum_{\beta < \alpha} \|\nabla \partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty L^2} \|\nabla \partial^{\alpha-\beta} W_\varepsilon\|_{L_v^2 \dot{H}_h^s} \|\partial^\alpha W_\varepsilon^h\|_{L_v^2 \dot{H}_h^s}. \end{aligned}$$

In this case, we define

$$C_{1,\varepsilon} \stackrel{\text{def}}{=} C \sum_{0 < \beta \leq \alpha} \|\partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty \dot{H}_h^{\frac{1}{2}}}^2 \|\partial^{\alpha-\beta} W_\varepsilon\|_{L_v^2 \dot{H}_h^{s+\frac{1}{2}}}^2$$

and

$$C_{2,\varepsilon} \stackrel{\text{def}}{=} C \|\nabla^h \partial^\alpha \bar{V}_\varepsilon^h\|_{L_v^\infty L^2}^2 (1 + \|\partial^\alpha \bar{V}_\varepsilon^h\|_{L_v^\infty L^2}^2) + C \sum_{0 < \beta \leq \alpha} \|\nabla \partial^\beta \bar{V}_\varepsilon^h\|_{L_v^\infty L^2} \|\nabla \partial^{\alpha-\beta} W_\varepsilon\|_{L_v^2 \dot{H}_h^s}$$

which as before are bounded in  $L^1(\mathbb{R}^+)$ , and we obtain, recalling (4.12),

$$\sum_{0 \leq \beta \leq \alpha} (**)_{\alpha,\beta} \leq \frac{1}{4} \|\nabla^h \partial^\alpha W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2 + C_{1,\varepsilon} + C_{2,\varepsilon} \|\partial^\alpha W_\varepsilon\|_{L_v^2 \dot{H}_h^s}^2.$$

The first part of the proposition is proved.

Now let us turn to the second part. As noted above, for all  $\alpha \in \mathbb{N}^3$ ,  $\partial^\alpha W_\varepsilon$  satisfies

$$\partial_t \partial^\alpha W_\varepsilon + \partial^\alpha (\bar{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon) - \Delta_h \partial^\alpha W_\varepsilon - \varepsilon^2 \partial_3^2 \partial^\alpha W_\varepsilon = -\partial^\alpha (\nabla^h Q_\varepsilon, \varepsilon^2 \partial_3 Q_\varepsilon).$$

Defining

$$g_\varepsilon \stackrel{\text{def}}{=} \bar{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon + (\nabla^h Q_\varepsilon, \varepsilon^2 \partial_3 Q_\varepsilon),$$

an energy estimate in  $L_v^2 \dot{H}_h^{-1}$  gives

$$\begin{aligned} (4.13) \quad \frac{1}{2} \|\partial^\alpha W_\varepsilon(t)\|_{L_v^2 \dot{H}_h^{-1}}^2 + \int_0^t \|\partial^\alpha W_\varepsilon(t')\|_{L^2}^2 dt' &\leq \frac{1}{2} \|\partial^\alpha W_{\varepsilon,0}\|_{L_v^2 \dot{H}_h^{-1}}^2 \\ &\quad + \int_0^t |\langle \partial^\alpha g_\varepsilon, \partial^\alpha W_\varepsilon \rangle_{L_v^2 \dot{H}_h^{-1}}(t')| dt'. \end{aligned}$$

We define  $K_\varepsilon(t) \stackrel{\text{def}}{=} \sup_{0 \leq t' \leq t} \|\partial^\alpha W_\varepsilon(t')\|_{\dot{H}_h^{-1}}$ , so that

$$\frac{1}{2} K_\varepsilon^2(t) \leq \frac{1}{2} \|\partial^\alpha W_{\varepsilon,0}\|_{L_v^2 \dot{H}_h^{-1}}^2 + K_\varepsilon(t) \int_0^t \|\partial^\alpha g_\varepsilon(t')\|_{L_v^2 \dot{H}_h^{-1}} dt'.$$

This implies that

$$(4.14) \quad \frac{1}{4} K_\varepsilon^2(t) \leq \frac{1}{2} \|\partial^\alpha W_{\varepsilon,0}\|_{L_v^2 \dot{H}_h^{-1}}^2 + \|\partial^\alpha g_\varepsilon\|_{L^1(\mathbb{R}^+, L_v^2 \dot{H}_h^{-1})}^2.$$

But according to (4.13) we know that

$$\int_0^t \|\partial^\alpha W_\varepsilon(t')\|_{L^2}^2 dt' \leq \frac{1}{2} \|\partial^\alpha W_{\varepsilon,0}\|_{L_v^2 \dot{H}_h^{-1}}^2 + K_\varepsilon(t) \int_0^t \|\partial^\alpha g_\varepsilon(t')\|_{L_v^2 \dot{H}_h^{-1}} dt',$$

so with (4.14) we infer that

$$\int_0^t \|\partial^\alpha W_\varepsilon(t')\|_{L^2}^2 dt' \lesssim \|\partial^\alpha W_{\varepsilon,0}\|_{L_v^2 \dot{H}_h^{-1}}^2 + \|\partial^\alpha g_\varepsilon\|_{L^1(\mathbb{R}^+, L_v^2 \dot{H}_h^{-1})}^2.$$

It remains to estimate  $\|\partial^\alpha g_\varepsilon\|_{L^1(\mathbb{R}^+, L_v^2 \dot{H}_h^{-1})}$ . As  $\bar{V}_\varepsilon^h$  is a divergence free vector field, we have

$$\begin{aligned} \|\partial^\alpha (\bar{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon)\|_{L^1(\mathbb{R}^+, L_v^2 \dot{H}_h^{-1})} &\leq \|\partial^\alpha (\bar{V}_\varepsilon^h \otimes W_\varepsilon)\|_{L^1(\mathbb{R}^+, L^2)} \\ (4.15) \qquad \qquad \qquad &\lesssim \sum_{0 \leq \beta \leq \alpha} \|\partial^\beta \bar{V}_\varepsilon^h\|_{L^2(\mathbb{R}^+, L_v^2 \dot{H}_h^{1/2})} \|\partial^{\alpha-\beta} W_\varepsilon\|_{L^2(\mathbb{R}^+, L_v^\infty \dot{H}_h^{1/2})} \end{aligned}$$

which gives the expected bound due to Proposition 4.1 (1) proved above. On the other hand, we recall that as computed in (4.2),

$$\Delta_h Q_\varepsilon - \varepsilon^2 \partial_3^2 Q_\varepsilon = \operatorname{div}_h (\bar{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon^h + \partial_3 (W_\varepsilon^3 \bar{V}_\varepsilon^h)).$$

so since  $(\Delta_h - \varepsilon^2 \partial_3^2)^{-1} \nabla_h \operatorname{div}_h$  and  $(\Delta_h - \varepsilon^2 \partial_3^2)^{-1} \varepsilon \partial_3 \operatorname{div}_h$  are zero-order Fourier multipliers, the same estimates give the expected a priori bound on  $(\nabla^h Q_\varepsilon, \varepsilon^2 \partial_3 Q_\varepsilon)$ , and the result follows.  $\square$

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